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Representations  
at a Root of Unity of  
 $q$ -Oscillators and  
Quantum Kac-Moody Algebras

by

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To my wife, Noriko, and  
the little baby she is now carrying

## Abstract

The subject of this thesis is quantum groups and quantum algebras at a root of unity. After an introductory chapter, I set up my notation in chapter 2. The rest of the thesis is presented in three parts.

In part I, quantum matrix groups and quantum enveloping algebras are discussed. In chapter 3, I present two well-known  $2 \times 2$  matrix quantum groups, including their coaction on the quantum plane and specialisations at a root of unity. Chapter 4 develops a quite detailed description of quantum enveloping algebras and their specialisation at an odd root of unity. The results from this chapter are required in part III.

Part II is devoted to certain deformations of the quantum mechanical oscillator algebra: so called  $q$ -oscillators. In chapter 5, a standard  $q$ -oscillator and its Fock module is described, including its specialisation at a root of unity. In chapter 6, original work [Pet93] on a new 2-parameter deformation of the oscillator algebra is presented and its representations at a root of unity are described.

Part III is concerned with infinite dimensional quantum groups. In chapter 7, the structure of an (untwisted) quantum affine Kac-Moody algebra is discussed. As in the classical case, it has both a Chevalley and a loop algebra presentation, which can be shown to be isomorphic using braid group and translation automorphisms. A quantum affine algebra has also a Heisenberg subalgebra: I describe its Fock modules and their unitarisability. Finally in chapter 8, I present original results [Pet94] on the specialisation of a quantum affine algebra at an odd root of unity. I prove that a quantum affine algebra at a root of unity has an infinite dimensional centre and construct the central elements corresponding to the real and imaginary roots. At the odd root of unity, some new infinite dimensional representations of the algebra are shown to exist.



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## CHAPTER 1

### Introduction

This thesis describes work I have done in the area of quantum groups. It is presented in three parts. Part I is largely concerned with simple quantum enveloping algebras and quantum groups. Part II is devoted to  $q$ -oscillators. Infinite dimensional quantum affine enveloping algebras are described in part III. The next chapter sets my notation for the rest of thesis.

I have tried to include references to the literature: the many omissions, that inevitably have occurred, should be put down to ignorance rather than ill-will on my part. Also the thesis has been written in a quite mathematical style, even though my background has been in physics: I hope that in the process I have not made it difficult to read for both mathematicians and theoretical physicists!

#### 1.1. Quantum groups

**1.1.1.** Quantum groups were first discovered by theoretical physicists to occur as symmetries of integrable 1+1-dimensional systems, particularly through the quantum inverse scattering mechanism [STF79]. The strong relation of quantum groups to the Yang-Baxter equation [Jim89] and transfer matrices in statistical mechanical models [Bax82] was also crucial in their development. It was Drinfeld who realised that mathematically quantum groups are Hopf algebras. These beginnings led to two major formulations of quantum groups that will be discussed in this thesis, although the first will only be described briefly.

Faddeev, Reshetikhin and Takhtajan [RTF90] developed the theory of matrix quantum groups. Drinfeld and Jimbo introduced 1-parameter deformations of universal enveloping algebras of semi-simple Lie algebras. Actually Drinfeld's and Jimbo's constructions, although similar in structure, have rather different origins. The motivation behind Drinfeld's approach is the quantisation of so-called Poisson-Lie groups. The approach of Jimbo, which will be the one followed in this thesis, is more Lie algebraic and leads to a much richer representation theory.

From a quite different direction, namely that of non-commutative geometry, Woronowicz [Wor87a] independently initiated the study of quantum groups from a  $C^*$ -algebra point of view. Manin [Man88] constructed quantum groups as the covariance symmetries of certain quantum (vector) spaces. The work of Woronowicz [Wor89] and Manin led to the field of non-commutative differential geometry of quantum groups [WZ90,

[SWZ92] and non-commutative spaces. Majid [Maj93] introduced braided groups and algebras.

In chapter 3, I discuss briefly some simple examples of matrix quantum groups, which co-act on the quantum plane. At a root of unity they have a large centre. At the end of the chapter, a 2-parameter deformed matrix quantum group [SWZ91] is mentioned.

**1.1.2.** More recently it has been realised that quantum groups occur as symmetries of a large number of systems in mathematical physics (see for instance [TEJ92, CFZ91]). Notably quantum enveloping algebras occur as symmetries in quantum spin chains and solvable lattice models [JM94], in two dimensional conformal field theory (see [Fuc92]) and in massive integrable systems [Smi92]. Quantum groups also play an important role in the theory of link invariants and knots [YG89]. One can say that by now the use of quantum groups in theoretical and mathematical physics has become so wide-spread that almost every day new preprints appear describing their structure and applications.

After Jimbo and Drinfeld introduced the quantum enveloping algebras of simple and affine Lie algebras, pure mathematicians (particularly Lie algebraists) gradually started to become very interested in these structures as a new development in Lie theory. Around 1990 many deep results on the representation theory of quantum enveloping algebras, particularly at a root of unity, appeared. Particularly notable is the work of Lusztig [Lus93] and of De Concini, Kac and Procesi. In their point of view, the specialisation of a quantum enveloping algebra at a  $p$ -th root of unity is a quantum version of the corresponding Lie algebra over a field of characteristic  $p$ . Lusztig's approach to quantum groups at a root of unity involves the specialisation of a subalgebra generated by divided powers with  $q$ -factorial numbers (a  $q$ -analogue of the  $\mathbb{Z}$ -form of a classical enveloping algebra of Chevalley, Kostant and Tits) — this does not give an enlarged centre at a root of unity. I will not have time to describe his interesting approach. Kashiwara and Lusztig independently introduced some important bases of quantum groups in [Kas90, Lus90a]

In chapter 4, I describe in some detail the structure of the quantum enveloping algebra of a simple Lie algebra. I introduce Lusztig's braid group automorphisms, that allow a construction of a basis of root vectors of the quantum enveloping algebra. Following De Concini and Kac, I apply the braid group automorphisms to construct the large centre of a quantum enveloping algebra at an odd root of unity. The Verma modules at a root of unity are reducible, they are 'replaced' by finite dimensional diagonal and (semicyclic) triangular modules. Cyclic modules also exist. The example of quantum enveloping algebra of  $\mathfrak{sl}_2$ , which first appeared in [KR83], is given and studied at a root of unity. Finally a 2-parameter deformation of  $\mathfrak{gl}_2$  is mentioned.

**1.1.3. Representations at a root of unity.** When the deformation parameter is not a root of unity, the representation theory of a quantum enveloping algebra is essentially the same [Lus88, Ros88] as that of the corresponding universal enveloping algebra of the simple Lie algebra.

One of the exciting aspects of quantum enveloping algebras at a root of unity is the possibility of new types of representations that have no classical analogue. The irreducible representations, in the case of an odd  $\ell$ -th root of unity, are parametrised by  $m$  continuous parameters (with  $m$  equal to the dimension of the underlying Lie

algebra) and have maximal dimension  $\ell^N$  (where  $N$  is the number of positive roots of the Lie algebra).

There are three distinct types of irreducible representations: nilpotent representations, semicyclic (semiperiodic) and cyclic (periodic) representations, but mixtures of these three types can also occur.

The nilpotent representations are deformations of classical representations with a highest and lowest weight, in which the positive and negative root vectors act nilpotently. They occur in 2d conformal field theories [AGGS90, GS90, Fuc92]

In a semicyclic representation, for each positive root, the associated positive or negative root vector (but not both) acts injectively. They are used in connection with lattice spin models and also relativistic solitons [GS92]. It is possible to modify certain nilpotent representations (of appropriate dimension) so that they become a semicyclic.

In a cyclic representation every root vector, corresponding to a positive or negative root, acts injectively. Cyclic modules have neither highest nor lowest weights, and their construction appears to be much more difficult, though considerable progress on this has been made (particularly for the case of cyclic modules of minimal dimension) [DJMM91a, AC91a, AC91b, CP91a, CP92, CP93, Sch93a, Sch93b]. Cyclic modules first occurred in the study of the 8-vertex model with the Sklyanin algebra [Skl83]. Cyclic representations occur [BKMS91, DJMM91b] in the context of the generalised Potts models. This work gave rise to new  $\mathcal{R}$ -matrices (solutions of the Yang-Baxter equation) which intertwine certain tensor products of minimal cyclic representations, which are the local state-spaces of the model, i.e. these  $\mathcal{R}$ -matrices are the Boltzmann weights of the model.

The difficult task of classifying the irreducible (cyclic) and indecomposable [PS90, AGL92] representations of a quantum enveloping algebra at a root of unity is still an open problem.

## 1.2. $q$ -oscillators

Since the invention of quantum mechanics, the quantum mechanical oscillator algebra has remained a key model in the theory of quantum physics and it is the basis of our understanding of quantum field theory and canonical quantisation. The oscillator algebra is also important in Lie algebra theory, since it allows the construction of realisations of Lie algebras. In retrospect then, it is perhaps not too surprising that  $q$ -analogues of the oscillator algebra were found in the context of quantum groups and from them  $q$ -oscillator realisations of quantum enveloping algebras could be constructed.

Chapter 5 has a detailed discussion of a typical  $q$ -oscillator algebra [Hay90] and its representation theory, and some comparisons with other  $q$ -oscillators. The algebra has a large centre at a root of unity. Irreducible finite dimensional quotients of the Fock modules exist. Semicyclic [SG91] and new cyclic modules are also constructed. Remarkably this algebra enjoys a unitary representation at even roots of unity. The quantum enveloping algebra of a classical simple Lie algebra can be bosonised with this algebra. I end this chapter with a new 2-parameter deformation of the oscillator algebra (compare [WV93]).

Chapter 6 is a ‘mathematised’ version of my work, published in [Pet93], describing the representations of a new quadratic (2-parameter)  $qr$ -deformation of the oscillator algebra, inspired by a 2-parameter deformation [Fai90] of  $\mathfrak{sl}_2$ . There are a number of improvements over the original work [Pet93]: for example the  $qr$ -oscillator algebra is considered at both odd and even roots of unity (rather than just even roots of unity). In both cases there exist irreducible finite dimensional quotients of its Fock modules. For real positive deformation parameters the Fock modules are unitarisable. Cyclic and semicyclic modules exist. Bosonisation maps for a number of well-known quadratic quantum algebras are constructed into the Heisenberg-Weyl subalgebra. The Heisenberg-Weyl subalgebra has a quantum group symmetry.

### 1.3. Quantum affine algebras

**1.3.1.** Affine Kac-Moody algebras [Kac90] are central to our present understanding of conformal field theory (Wess-Zumino-Witten models), solvable lattice models, and low dimensional integrable systems. From the point view of Lie theory, they are also a generalisation of the theory of simple (finite) Lie algebras. They are characterised by Cartan matrices of affine type. To each affine Cartan matrix there is a quantum enveloping algebra (due to Jimbo and Drinfeld), called a quantum affine algebra.

Quantum affine algebras arise as symmetries of quantum spin chains, solvable vertex and lattice models. In [DFJMN93] the XXZ-spin chain model was studied in the thermodynamic limit. Using the representation theory of quantum affine algebras,  $q$ -vertex operators [FR92] and corner transfer matrix methods, the authors were able to diagonalise the XXZ-Hamiltonian and find its vacuum vectors. After they calculated correlation functions for the model [JMMN92] as traces of bosonised [FJ88] level 1 quantum affine algebra vertex operators, a large number of authors studied the bosonisations [Ber89, Mat93, Bou93, Kim92, AOS93, BW93] of quantum affine algebras further and  $q$ -analogues of the Wakimoto construction were obtained. Quantum affine algebras are also symmetries of some 2 dimensional integrable quantum field theories. For example quantum affine algebras are important in Toda theory [BL91, FF93]. A further interest in quantum affine algebras is in the programme to try to develop a more geometrical understanding (“ $q$ -conformal field theory”) [FR92, Mat92] and to better understand its relation to 2d integrable field theories [Luk93].

The representation theory of quantum affine algebras (for generic  $q$ ) is quite well understood. The results of Lusztig [Lus88] and Rosso [Ros88], describing the equivalence of the irreducible representations of a finite quantum group (at generic  $q$ ) to the irreducible representations of the corresponding simple Lie algebra, extend [Lus93, 33] to the case of a quantum affine Kac-Moody algebra. In analogy with the classical results (see [Kac90]), it has been shown [ZG93] that (a certain category of) integrable modules over a quantum affine algebra (which includes all highest weight representations) are completely reducible and that every irreducible integrable highest weight module over a non-twisted quantum affine algebra with  $q \in \mathbb{R}_{>0}$  is unitary. Chari and Pressley [CP91b] have classified the irreducible finite dimensional evaluation representations (IFDRs) of the quantum affine algebra of  $\mathfrak{sl}_2$  and their tensor products.

In chapter 7, I describe the quantum affine algebra of a (untwisted) affine Kac-

Moody algebra and a recent extension of the braid group automorphisms [Bec93], which allowed an explicit proof that Drinfeld's loop algebra presentation is isomorphic to the usual one. Fock modules of the Heisenberg subalgebra can be constructed. They are unitarisable when the deformation parameter is positive real.

**1.3.2. Quantum affine algebras at a root of unity.** The structure of a quantum affine algebra at a root of unity is not well understood. Infinite dimensional representations of a quantum affine algebra at a root of unity at non-zero level, i.e. representations which are not evaluation representations or loop modules, have not been studied in the literature.

Finite dimensional minimal cyclic representations of quantum affine algebras occur [BKMS91, DJMM90, DJMM91c] in the context of the generalised Potts models. These representations are essentially evaluation map representations of the quantum affine algebra at a root of unity onto the cyclic (periodic) representation of the corresponding finite quantum group.

In chapter 8, I present my results [Pet94] on a quantum affine algebra at a root of unity. I show that a quantum affine algebra has an infinite dimensional centre at a root of unity. In particular I prove that at an odd  $\ell$ -th root of unity every real root vector in the algebra raised to the  $\ell$ -th power lies in the centre of the algebra. Also in the Heisenberg subalgebra, every imaginary root vector, whose mode number is a multiple of  $\ell$  is also central. Nevertheless a quantum affine algebra at a root of unity is infinite dimensional over its centre, and its modules with nonzero level (mod  $\ell$ ) are necessarily infinite dimensional.

An interesting problem would be to try to construct level 1 (and higher) vertex operators at a  $\ell$ -th root of unity. I hope that my results on a quantum affine algebra at a root of unity may help with this problem and find applications in solvable lattice models and integrable systems. They may for example have some relevance to the massless phase of the XXZ spin chain model (for which  $|q| = 1$ ).

## 1.4. Summary of original results

Part I of the thesis is essentially all review. Most of the results in chapter 5 are well-known, though I believe proposition 5.2.8, lemma 5.3.4, lemma 5.7.2, lemma 5.7.3 and section 5.8 are new. Chapter 6 is original work. Most of the results of chapter 7 have appeared already in the literature or are natural generalisations of classical results, but I have not seen lemma 7.2.11, lemma 7.5.11 and most of section 7.6 before. Chapter 8 is original work, the most important results being proposition 8.2.2, lemma 8.2.6, proposition 8.3.5, lemma 8.3.7 and proposition 8.5.6.

## 1.5. Acknowledgements and thanks

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## CHAPTER 2

### Preliminaries

Let  $\mathbb{Z}$  denote the set of integers,  $\mathbb{R}$  the real numbers and  $\mathbb{C}$  the field of complex numbers. Let  $\mathbb{Z}^\times$ ,  $\mathbb{R}^\times$  and  $\mathbb{C}^\times$  denote the corresponding sets with  $\{0\}$  removed. For  $m, n \in \mathbb{Z}$ , let  $[m, n]$  denote the set of integers in the range from  $m$  to  $n$  (empty if  $m > n$ ). Let  $\mathbb{N}$  denote the natural numbers (non-negative integers). Let  $\mathbb{Z}_{>0}$  denote the positive integers and let  $\mathbb{Z}_{<0}$  denote the negative integers. Let  $\mathbb{R}_{>0}$  be the positive real numbers.

I start by recalling the definitions of some elementary algebraic structures, to set up my notation.

#### 2.1. Groups, Rings and Fields

**DEFINITION 2.1.1 (GROUP).** A group  $(G, \cdot)$  is a set  $G$  with an operation  $\cdot$ , such that (i) there is an identity element  $e$  in  $G$

$$e \cdot g = g \quad \text{and} \quad g \cdot e = g \quad (\forall g \in G);$$

(ii) the operation  $\cdot$  is closed

$$g \cdot g' \in G \quad (\forall g, g' \in G);$$

(iii) the operation is associative

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3 \quad (\forall g_1, g_2, g_3 \in G);$$

(iv) for every element of  $g \in G$  there exists an inverse element  $g^{-1}$

$$g \cdot g^{-1} = e \quad \text{and} \quad g^{-1} \cdot g = e \quad (\forall g \in G).$$

**DEFINITION 2.1.2 (RING).** A ring  $(R, +, \cdot)$  is a set  $R$  with an addition operation  $+$  and a multiplication operation  $\cdot$ , such that (i)  $(R, +)$  is a commutative group, (ii) the multiplication is associative and (iii) the multiplication and addition are distributive:

$$(a + b) \cdot c = a \cdot c + b \cdot c \quad \text{and} \quad a \cdot (b + c) = a \cdot b + a \cdot c \quad (\forall a, b, c \in R).$$

The identity element of  $(R, +)$  is usually denoted 0. If a ring  $R$  has a unity element 1 (such that  $1 \cdot x = x = x \cdot 1$  ( $x \in R$ )) then  $R$  is called a ring with unity  $(R, +, \cdot, 1)$ .

A ring is said to be *commutative* if  $ab = ba \quad \forall a, b \in R$ .

Let  $(R, +, \cdot, 1)$  be a ring with unity. An element in  $R$  is called a unit if it has both a left and a right multiplicative inverse. A ring in which every element, except 0, is a unit is called a division ring. A commutative division ring is called a *field*.

NOTATION. From here on, unless otherwise stated, a field will always assumed to be of characteristic 0.

EXAMPLE 2.1.3. Let  $R$  be a ring. The ring  $R[x]$  of polynomials in an indeterminant  $x$  with coefficients valued in  $R$  is an example of a commutative infinite dimensional ring. This can be extended to the commutative ring  $R[x_1, x_2, \dots, x_n]$  of polynomials in  $n$  indeterminants with coefficients in  $R$ . Often it is also desirable to consider the ring  $R[x, x^{-1}]$  of polynomials in an indeterminant  $x$  and its inverse  $x^{-1}$ .

From a commutative ring  $R$  with no zero divisors (equivalently an entire ring  $R$  or integral domain  $R$ ), a quotient field  $Q(R)$  can be constructed.

PROPOSITION 2.1.4 (QUOTIENT FIELD). *Let  $R$  be a commutative ring with no zero divisors and let  $R' := R \setminus \{0\}$ . Consider all pairs  $(a, b) \in R \times R'$ . Then  $(a, b) \sim (c, d)$ , if  $a \cdot d = b \cdot c$ , defines an equivalence relation on  $R \times R'$ . Define  $Q(R) := R \times R' / \sim$  and denote the equivalence class containing  $(a, b)$  by  $[(a, b)]$ . The addition and product in  $Q(R)$  defined as*

$$[(a, b)] \cdot [(c, d)] := [(ac, bd)] \quad \text{and} \quad [(a, b)] + [(c, d)] := [(ad + bc, bd)],$$

give  $Q(R)$  the structure of a field, the quotient field of  $R$ .

PROOF. See for instance [Fra82, chapter 26].  $\square$

EXAMPLE 2.1.5. The quotient field of a ring of polynomials  $R[x]$ : eg  $\mathbb{C}(q) := Q(\mathbb{C}[q, q^{-1}])$  is the quotient field of the ring of polynomials  $\mathbb{C}[q, q^{-1}]$ .

## 2.2. Modules and Vector spaces

Later when I come to consider the representations of quantum groups and quantum algebras, the notation of modules and vector spaces will be important.

DEFINITION 2.2.1 (MODULE). Let  $R$  be a ring. A *left-module*  $(M, +, R, \cdot)$  over  $R$  (or left  $R$ -module) is an abelian group  $(M, +)$  and an operation of  $(R, \cdot)$  on  $M$  ( $R \times M \mapsto M$ ) such that

$$(a + b) \cdot u = a \cdot u + b \cdot u \quad \text{and} \quad a \cdot (u + v) = a \cdot u + a \cdot v \quad (a, b \in R, u, v \in M).$$

A module over a field  $k$  is called a *vector space* ( $k$ -vector space).



### 2.3. Associative Algebras

NOTATION. Let  $U$  be a space (set). Denote by  $\text{id}$  the trivial map  $\text{id} : U \rightarrow U$ , which maps  $u \mapsto u$  ( $u \in U$ ). When necessary, I use a subscript  $\text{id}_U$  to emphasise the space on which the map is acting.

DEFINITION 2.3.1 (ALGEBRA). Let  $R$  be a commutative ring. An *algebra*  $(\mathcal{A}, +, R, \cdot)$  over  $R$  (or  $R$ -algebra) is an  $R$ -module  $\mathcal{A}$  with a bilinear product map  $m : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ . An algebra is said to be associative if

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad (x, y, z \in \mathcal{A}).$$

If  $\mathcal{A}$  has an element  $\mathbf{1}$  such that  $x \cdot \mathbf{1} = x = \mathbf{1} \cdot x$  ( $x \in \mathcal{A}$ ), then the algebra is called an  $R$ -algebra with unity, and there is a natural map  $\mu : R \rightarrow \mathcal{A}$ , which maps  $\mu : a \mapsto a \cdot \mathbf{1}$  ( $a \in R$ ) called the unity map, by definition it satisfies

$$m \circ (\mu \times \text{id})(a, x) = a \cdot x = m \circ (\text{id} \times \mu)(x, a) \quad (a \in R, x \in \mathcal{A}).$$

NOTATION. From here on, unless otherwise stated, an algebra will always be assumed to be associative.

REMARK 2.3.2. Let  $(\mathcal{A}, +, R, \cdot)$  be an associative algebra. The bilinear product of the algebra maps  $m : (x, y) \mapsto x \cdot y$ . Then the associativity of algebra can be expressed as

$$m \circ (m \times \text{id})(x, y, z) = m \circ (\text{id} \times m)(x, y, z) \quad (x, y, z \in \mathcal{A}).$$

EXAMPLE 2.3.3. Let  $V$  be a vector space. Denote by  $\text{End}(V)$  the set of all linear maps  $V \rightarrow V$ , the algebra of (linear) endomorphisms of  $V$ . Given a basis of  $V$ , the elements of  $\text{End}(V)$  can be realised as matrices.

DEFINITION 2.3.4 (FREE ALGEBRA). Let  $X := \{x_1, x_2, \dots, x_n\}$  be a set of  $n$  distinct letters. Consider the set of finite (noncommutative) monomials in the elements of  $X \cup 1$  ("strings of letters in the alphabet  $X$ "). The set of linear combinations of these monomials with coefficients in a ring  $R$  (noncommutative polynomials)  $R[[x_1, x_2, \dots, x_n]]$  is naturally an abelian group  $(R[[x_1, x_2, \dots, x_n]], 1, +)$ . It can be given the structure of an associative unital algebra  $(R[[X]], +, 1, \cdot)$  by defining the multiplication of any pair of monomials  $x_{i_1} \cdots x_{i_r}$  and  $x_{j_1} \cdots x_{j_s}$  ( $i_1, \dots, i_r \in X, j_1, \dots, j_s \in X$  and  $r, s \in \mathbb{Z}_{>0}$ ) in an obvious way to be

$$\begin{aligned} (x_{i_1} \cdots x_{i_r}) \cdot (x_{j_1} \cdots x_{j_s}) &:= x_{i_1} \cdots x_{i_r} \cdot x_{j_1} \cdots x_{j_s}, \\ (x_{i_1} \cdots x_{i_r}) \cdot 1 &:= (x_{i_1} \cdots x_{i_r}), \\ 1 \cdot (x_{i_1} \cdots x_{i_r}) &:= (x_{i_1} \cdots x_{i_r}). \end{aligned}$$

This  $R$ -algebra is called the *free  $R$ -algebra* on the generators in the set  $X$ .

DEFINITION 2.3.5 (SUBALGEBRA). Let  $\mathcal{A}$  be an  $R$ -algebra, a subset  $\mathcal{B} \subset \mathcal{A}$  is called a  $R$ -subalgebra of  $\mathcal{A}$ , if  $\mathcal{B}$  forms a (closed) algebra in  $\mathcal{A}$ .

REMARK 2.3.6. Let  $S$  be a subring of  $R$  and let  $\mathcal{A}$  be a  $R$ -algebra, then there is another type of subalgebra possible: an  $S$ -subalgebra of  $\mathcal{A}$ .

DEFINITION 2.3.7. Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $R$ -algebras. A  $R$ -linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}$ , is called an  $R$ -algebra homomorphism if:

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y) \quad \text{and} \quad \phi(x + y) = \phi(x) + \phi(y) \quad (x, y \in \mathcal{A}).$$

If the homomorphism is injective ('one-to-one') and surjective ('onto'), the map is called an isomorphism and the algebras are said to be isomorphic.

DEFINITION 2.3.8 (IDEAL). Let  $\mathcal{A}$  be an algebra, a subalgebra  $\mathcal{I}$  of  $\mathcal{A}$  is called a left (respectively right) ideal, if  $\forall x \in \mathcal{I}, \mathcal{A} \cdot x \subseteq \mathcal{I}$  (respectively  $x \cdot \mathcal{A} \subseteq \mathcal{I}$ ). A subalgebra which is both a left and a right ideal, is called a two sided ideal. If  $\mathcal{I} \neq \mathcal{A}$ , then  $\mathcal{I}$  is a proper ideal of  $\mathcal{A}$ . If  $\mathcal{I} \neq \{0\}$ , then  $\mathcal{I}$  is a nontrivial ideal of  $\mathcal{A}$ .

NOTATION. From here on, unless otherwise stated, an ideal of an algebra will always be a proper, non-trivial, two sided ideal.

LEMMA 2.3.9. Let  $\mathcal{A}$  be an  $R$ -algebra, with a (two sided) ideal  $\mathcal{I}$ . There is a canonical homomorphism  $\phi : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$ , called the projection map. It follows that the quotient  $\mathcal{A}/\mathcal{I}$  is an algebra (the quotient algebra of  $\mathcal{A}$  by  $\mathcal{I}$ ).

PROOF. Consider elements  $x, x', y, y' \in \mathcal{A}$ , such that  $x \neq x'$  and  $y \neq y'$ , but  $\phi(x) = \phi(x')$  and  $\phi(y) = \phi(y')$ . Then I wish to check that the addition and multiplication are well defined in  $\mathcal{A}/\mathcal{I}$ , i.e.

$$\phi(x + y) = \phi(x' + y') \quad \text{and} \quad \phi(x \cdot y) = \phi(x' \cdot y').$$

Let  $a = x - x'$  and  $b = y - y'$ . Clearly  $a, b \in \mathcal{I}$ . Then  $x + y = x' + y' + a + b$  and  $x \cdot y = x' \cdot y' + x' \cdot b + a \cdot y'$ . But  $a + b, x' \cdot b, a \cdot y' \in \mathcal{I}$  and the lemma is proven.  $\square$

LEMMA 2.3.10. Let  $\mathcal{A} := k[[x_1, \dots, x_n]]$  be a free  $k$ -algebra (on  $n$  generators). Let  $\{r_1, \dots, r_m\}$  be  $m$  distinct elements in  $\mathcal{A}$  and let  $\mathcal{J} := \text{Span}_k(r_1, \dots, r_m)$ . Then  $\mathcal{I} := \mathcal{A} \cdot \mathcal{J} \cdot \mathcal{A}$  is a two-sided ideal in  $\mathcal{A}$ . Therefore the quotient  $\mathcal{A}/\mathcal{I}$  is an algebra.

DEFINITION 2.3.11 (CENTRE). Let  $\mathcal{A}$  be an  $R$ -algebra. The centre  $\mathcal{Z}(\mathcal{A})$  of  $\mathcal{A}$  is defined to be the set  $\{x \in \mathcal{A} \mid x \cdot y = y \cdot x, \forall y \in \mathcal{A}\}$ .

## 2.4. Combining Vector spaces and Tensor algebras

DEFINITION 2.4.1 (CARTESIAN PRODUCT). Let  $V$  and  $W$  be two spaces. Then their Cartesian product is defined to be  $V \times W := \{(v, w) \mid v \in V, w \in W\}$ .

Let  $V$  and  $W$  be  $R$ -modules. Two types of  $R$ -module structure can be naturally defined on their Cartesian product  $V \times W$ .

**2.4.2. Direct sum.** Let  $V$  and  $W$  be  $R$ -modules and  $V \times W$  be their Cartesian product.  $V \times W$  can be endowed with the structure of a  $R$ -module  $V \oplus W$ , the *direct sum* of  $V$  and  $W$ , by defining

$$\begin{aligned} a \cdot (v, w) &:= (a \cdot v, a \cdot w) \\ (v, w) + (v', w') &:= (v + v', w + w') \end{aligned} \quad a \in R, v, v' \in V, w, w' \in W.$$

**2.4.3. Tensor product.** Let  $R$  be a ring. Let  $V$  and  $W$  be  $R$ -modules. Let  $M$  be the free module generated over  $R$  by the set  $V \times W$ . Let  $N$  be the submodule of  $M$  defined by

$$N := \{(v + v', w) - (v, w) - (v', w), (v, w + w') - (v, w) - (v, w'), \\ (a \cdot v, w) - a(v, w), (v, a \cdot w) - a(v, w) \mid \forall v, v' \in V, w, w' \in W, a \in R\}.$$

Then the canonical bilinear map  $\phi : V \times W \rightarrow M/N := V \otimes W$  defines the  $R$ -tensor product of  $V$  and  $W$ . Let  $v \in V$  and  $w \in W$ . Denote  $\phi(v, w)$  by  $v \otimes w$ . By construction the following identities hold in  $V \otimes W$

$$\begin{aligned} (a \cdot v) \otimes w &\equiv a \cdot (v \otimes w) \equiv v \otimes (a \cdot w), \\ (v + v') \otimes w &\equiv v \otimes w + v' \otimes w, & a \in R, v, v' \in V, w, w' \in W \\ v \otimes (w + w') &\equiv v \otimes w + v \otimes w'. \end{aligned}$$

The tensor product of two modules is a module and the tensor product over a field  $k$  of two  $k$ -vector spaces is a vector space.

Having taken the tensor product  $U \otimes V$  of two  $R$ -modules  $U$  and  $V$ , the process can be repeated: I can take the tensor product of  $U \otimes V$  with another  $R$ -module  $W$  to create  $(U \otimes V) \otimes W$  (or  $W \otimes (U \otimes V)$ ).

**PROPOSITION 2.4.4 (ASSOCIATIVITY).** *Let  $U$ ,  $V$  and  $W$  be  $k$ -vector spaces. There is a unique isomorphism  $(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ , which maps*

$$(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$$

**PROOF.** See for example [Lan93, chapter XVI].  $\square$

**LEMMA 2.4.5 (TWIST).** *Let  $V$  and  $W$  be  $k$ -vector spaces. There is a unique isomorphism  $V \otimes W \rightarrow W \otimes V$ ,  $v \otimes w \mapsto w \otimes v$ .*

**LEMMA 2.4.6 (TENSOR PRODUCT OF ALGEBRAS).** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $k$ -algebras. Consider then their tensor product  $\mathcal{A} \otimes \mathcal{B}$  as vector spaces. This vector space can be given the structure of a  $k$ -algebra by defining the product in  $\mathcal{A} \otimes \mathcal{B}$  to be*

$$(v_1 \otimes w_1) \cdot (v_2 \otimes w_2) := (v_1 \cdot v_2) \otimes (w_1 \cdot w_2).$$

A very important construction among associative algebras is that of the tensor algebra of a vector space. The universality of tensor algebras is essentially contained in the following fact: every (finitely generated) associative algebra is the quotient of the tensor algebra of some (finite dimensional) vector space.

**DEFINITION 2.4.7 (TENSOR ALGEBRA).** Let  $V$  be a  $k$ -vector space. For each integer  $n \in \mathbb{Z}_{>0}$  define  $T^n(V) := \bigotimes_{i=1}^n V$ . Define  $T^0(V) := k$ . Because the tensor product is associative (proposition 2.4.4), there is a bilinear map  $T^r(V) \times T^s(V) \rightarrow T^{r+s}(V)$ . Using this I can give the space

$$T(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n} \equiv k \oplus (V) \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

the structure of a  $k$ -algebra the *tensor algebra*  $T(V)$  of  $V$ . The product in  $T(V)$  is again denoted  $\otimes$ .

LEMMA 2.4.8. (i) *Let  $T(V)$  be the tensor algebra of a  $k$ -vector space  $V$ .  $T(V)$  is an associative, infinite dimensional  $k$ -algebra.*

(ii) *Let  $W$  be a  $k$ -vector space such that  $\dim(W) = \dim(V)$ . Then  $T(V)$  and  $T(W)$  are isomorphic as  $k$ -algebras.*

LEMMA 2.4.9. *Let  $k[[x_1, x_2, \dots, x_n]]$  be a free  $k$ -algebra with unity and let  $V$  be an  $n$ -dimensional  $k$ -vector space. Then  $k[[x_1, x_2, \dots, x_n]]$  is isomorphic as a  $k$ -algebra to  $T(V)$ .*

Let  $\{v_1, v_2, \dots, v_n\}$  be a linear basis of  $V$ . Then the map

$$\begin{aligned} k[[x_1, x_2, \dots, x_n]] &\rightarrow T(V), \\ x_i &\mapsto v_i, \end{aligned}$$

is an  $k$ -algebra isomorphism.

NOTATION. Let  $\mathcal{A}$  be a  $k$ -algebra. I will denote the two-sided ideal  $\mathcal{I}$  in  $\mathcal{A}$  (freely) generated over  $k$  by elements  $\{r_1, \dots, r_n\} \subset \mathcal{A}$  (cf. 2.3.10) by the notation

$$\langle r_1, \dots, r_n \rangle := \mathcal{I} \equiv \mathcal{A} \cdot \text{Span}(\{r_1, \dots, r_n\}) \cdot \mathcal{A}.$$

REMARK 2.4.10. A important problem when considering the quotient of a tensor or free algebra by a two-sided ideal is whether the quotient algebra is associative or not.

I briefly recall the construction of the symmetric algebra from the tensor algebra of a vector space.

PROPOSITION 2.4.11 (SYMMETRIC ALGEBRA). *Let  $V$  be a  $k$ -vector space and  $T(V)$  its tensor algebra. The elements  $\{x \otimes y - y \otimes x \mid x, y \in V\}$  generate an ideal in  $T(V)$ ,  $I_{\text{sym}} := \langle x \otimes y - y \otimes x \mid x, y \in V \rangle$ . The quotient algebra  $S(V) := T(V)/I_{\text{sym}}$  is a commutative infinite dimensional algebra, the symmetric algebra of  $V$ .*

PROOF. See for instance [Hum72, chapter V].  $\square$

REMARK 2.4.12. Let  $V$  be a  $n$ -dimensional  $k$ -vector space. Quotienting  $T(V)$  by an anti-symmetrising ideal, the exterior (grassmann) algebra

$$A(V) := T(V)/\langle a \otimes b + b \otimes a \mid a, b \in V \rangle,$$

on  $V$  is constructed.  $A(V)$  is a  $2^n$  dimensional associative  $k$ -algebra.

### 2.5. Lie algebras and Universal enveloping algebras

DEFINITION 2.5.1 (LIE ALGEBRA). A *Lie algebra*  $(\mathfrak{g}, [\cdot, \cdot])$  is a  $k$ -algebra with a  $k$ -bilinear product map  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying

$$\begin{aligned} (\text{antisymmetry}) \quad & [x, x] = 0 \quad (x \in \mathfrak{g}), \\ (\text{Jacobi identity}) \quad & [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad (x, y, z \in \mathfrak{g}). \end{aligned}$$

REMARK 2.5.2. The antisymmetry property  $[x, x] = 0$  ( $x \in \mathfrak{g}$ ) implies that  $[x, y] = -[y, x]$ . This is easily seen by considering  $[x + y, x + y]$ .

LEMMA 2.5.3. Let  $(\mathcal{A}, \cdot, k)$  be an associative  $k$ -algebra. Defining the commutator  $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$

$$[x, y] := x \cdot y - y \cdot x \quad (x, y \in \mathcal{A}),$$

then  $(\mathcal{A}, [\cdot, \cdot])$  is a Lie algebra.

PROOF. Antisymmetry follows immediately from the definition of the commutator and the Jacobi identity from the associativity of  $\mathcal{A}$ .  $\square$

DEFINITION 2.5.4 (LIE SUBALGEBRA). Let  $\mathfrak{g}$  be a Lie algebra. Let  $\mathfrak{h}$  be a linear subspace of  $\mathfrak{g}$ . If  $\mathfrak{h}$  itself forms a closed Lie algebra, then  $\mathfrak{h}$  is called a Lie subalgebra of  $\mathfrak{g}$ .

EXAMPLE 2.5.5. Let  $n \in \mathbb{Z}_{>0}$ . Consider the  $k$ -algebra of  $n \times n$ -matrices  $\mathcal{M}_n$  (with entries valued in  $k$ ). Then  $(\mathcal{M}_n, [\cdot, \cdot])$  is a Lie algebra, usually denoted  $\mathfrak{gl}_n(k)$ . The subset of  $\mathcal{M}_n$  of traceless matrices, forms a Lie subalgebra, denoted  $\mathfrak{sl}_n(k)$ .

The standard way of analysing the structure of Lie algebras is by consideration of their ideals.

DEFINITION 2.5.6 (IDEAL). Let  $\mathfrak{g}$  be a Lie algebra and  $\mathfrak{h}$  an Lie subalgebra of  $\mathfrak{g}$ . If  $[u, x] \in \mathfrak{h}$  ( $u \in \mathfrak{h}$  and  $x \in \mathfrak{g}$ ), then  $\mathfrak{h}$  is called an ideal of  $\mathfrak{g}$ .

DEFINITION 2.5.7 (SIMPLE). Let  $\mathfrak{g}$  be a Lie algebra, which as a vector space has dimension greater than 1. The Lie algebra  $\mathfrak{g}$  is called *simple*, if it has no (proper, non-trivial) ideals.

LEMMA 2.5.8 (QUOTIENT ALGEBRA). Let  $\mathfrak{g}$  be a Lie algebra and  $\mathfrak{h}$  an ideal in  $\mathfrak{g}$ . Then the quotient algebra  $\mathfrak{g}/\mathfrak{h}$  is a Lie algebra.

LEMMA 2.5.9 (DERIVED LIE ALGEBRA). Let  $\mathfrak{g}$  be a Lie algebra. The subset  $\mathfrak{g}' := \{[x, y] \mid x, y \in \mathfrak{g}\} =: [\mathfrak{g}, \mathfrak{g}]$  forms an ideal in  $\mathfrak{g}$ .

PROOF. Clearly  $\mathfrak{g}'$  is a Lie subalgebra of  $\mathfrak{g}$ , since

$$[[x, y], [x', y']] \in \mathfrak{g}' \quad x, x', y, y' \in \mathfrak{g}.$$

That  $\mathfrak{g}'$  is an ideal of  $\mathfrak{g}$ , follows since  $[\mathfrak{g}, \mathfrak{g}'] \subseteq \mathfrak{g}'$   $\square$

It is well-known that universal enveloping algebras play an important role in Lie algebra and Lie group theory, specially in their representation theory (see for example [Hum72]).

**PROPOSITION 2.5.10 (UNIVERSAL ENVELOPING ALGEBRA).** *Let  $(\mathfrak{g}, [\cdot, \cdot], +)$  be a Lie algebra. Let  $T(\mathfrak{g})$  be the tensor algebra of the  $k$ -vector space  $(\mathfrak{g}, +)$ . Consider then the following ideal in  $T(\mathfrak{g})$*

$$I_{\mathfrak{g}} := \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle$$

*The quotient algebra  $\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g})/I_{\mathfrak{g}}$  is called the universal enveloping algebra.  $\mathcal{U}(\mathfrak{g})$  is an associative infinite dimensional algebra with unity.*

## 2.6. Representations and Reducibility

**DEFINITION 2.6.1 (REPRESENTATION).** Let  $V$  be a  $k$ -vector space and let  $\mathcal{A}$  be a  $k$ -algebra. If  $\pi : \mathcal{A} \rightarrow \text{End}(V)$  is a  $k$ -algebra homomorphism:

$$\pi(x \cdot y) \cdot v = \pi(x) \cdot \pi(y) \cdot v, \quad x, y \in \mathcal{A}, v \in V,$$

then the pair  $(V, \pi)$  is called a (linear) representation of  $\mathcal{A}$  on  $V$ . The vector space  $V$  carrying the representation  $\pi$  of  $\mathcal{A}$  is also called an  $\mathcal{A}$ -module.

**DEFINITION 2.6.2 (INTERTWINER).** Let  $V$  and  $W$  be  $\mathcal{A}$ -modules. A vector space homomorphism  $\phi : V \rightarrow W$  is called an  $\mathcal{A}$ -module homomorphism, if  $\phi$  is such that:

$$\phi(\pi_V(a) \cdot v) = \pi_W(a) \cdot \phi(v), a \in \mathcal{A}, v \in V.$$

Such a map is also called an intertwining map (intertwiner) from  $V$  to  $W$ . If further  $\phi$  is an isomorphism of vector spaces, then  $\phi$  is called an  $\mathcal{A}$ -module isomorphism and the two modules  $V$  and  $W$  are *equivalent* as  $\mathcal{A}$ -modules.

**DEFINITION 2.6.3 (SUBMODULE).** Let  $V$  be an  $\mathcal{A}$ -module. If  $V$  contains a proper linear subspace  $W$ , such that  $W$  is closed under the action of  $\mathcal{A}$  (i.e.  $\pi(\mathcal{A}) \cdot W \subseteq W$ ) then  $W$  is said to carry a *subrepresentation* of  $\mathcal{A}$  and  $W$  is a  $\mathcal{A}$ -submodule of  $V$ .

**DEFINITION 2.6.4 (IRREDUCIBLE).** An  $\mathcal{A}$ -module is called *irreducible*, if it has no (proper, non-trivial) submodules, otherwise it is called *reducible*.

**DEFINITION 2.6.5 (COMPLETELY REDUCIBLE).** An  $\mathcal{A}$ -module is called *completely reducible*, if it is equivalent to a direct sum of irreducible  $\mathcal{A}$ -modules.

**DEFINITION 2.6.6 (INDECOMPOSABLE).** An  $\mathcal{A}$ -module is called *indecomposable*, if it is not equivalent to a direct sum of (two or more)  $\mathcal{A}$ -modules, otherwise the module is called *decomposable*.

An irreducible  $\mathcal{A}$ -module is always indecomposable, but in general there will exist indecomposable  $\mathcal{A}$ -modules which are not irreducible. Of course the full reduction of an indecomposable representation gives an irreducible one.

## 2.7. Hopf Algebras

I come now to the definition of some of the key objects, examples of which will be discussed in detail throughout this thesis. Earlier I wrote down the definition of an algebra. I define next a dual structure. It is dual in the sense of category theory and because it basically defines a structure with an “unmultiplication” map.

NOTATION. Let  $V$  be an  $R$ -module. The map  $\sigma : V \otimes V \rightarrow V \otimes V$ , which maps  $x \otimes y \mapsto y \otimes x$  ( $x, y \in V$ ), is called the twist map.

DEFINITION 2.7.1 (COALGEBRA). Let  $R$  be a ring. A *coalgebra*  $(\mathcal{C}, +, R, \Delta)$  over  $R$  ( $R$ -coalgebra) is a  $R$ -module  $\mathcal{C}$  with a linear map  $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$  called the coproduct (or comultiplication map). If

$$(\text{id} \times \Delta) \circ \Delta(x) = (\Delta \times \text{id}) \circ \Delta(x) \quad (\forall x \in \mathcal{C})$$

then the coproduct  $\Delta$  and the coalgebra are called coassociative. If there exists a map  $\epsilon : \mathcal{C} \rightarrow R$  such that

$$(\epsilon \times \text{id}) \circ \Delta(x) = x = (\text{id} \times \epsilon) \circ \Delta(x) \quad (\forall x \in \mathcal{C}),$$

then the map  $\epsilon$  is called a counit map and the coalgebra is called a coalgebra with counit  $(\mathcal{C}, +, R, \Delta, \epsilon)$ . A coalgebra  $(\mathcal{C}, \Delta)$  is called *cocommutative* if  $\sigma \circ \Delta(x) = \Delta(x)$  ( $\forall x \in \mathcal{C}$ ).

LEMMA 2.7.2. Let  $(\mathcal{A}, +, m; k)$  be a (finite)  $k$ -algebra and let  $(\mathcal{A}^*, +; k)$  be the  $k$ -vector space dual to  $(\mathcal{A}, +; k)$ , with the dual pairing  $\langle \cdot, \cdot \rangle : \mathcal{A}^* \otimes \mathcal{A} \rightarrow k$ . Then the dual pairing induces a coalgebra structure  $(\mathcal{A}^*, +, \Delta; k)$  on  $\mathcal{A}^*$

$$\langle \Delta^*(x^*), x \otimes y \rangle := \langle x^*, m(x \otimes y) \rangle \quad (x^* \in \mathcal{A}^*; x, y \in \mathcal{A}).$$

Many examples of coalgebras are also at the same time algebras. This motivates the following definition.

DEFINITION 2.7.3 (BIALGEBRA). Let  $R$  be a ring and  $\mathcal{B}$  an  $R$ -algebra.  $\mathcal{B}$  has the structure of a *bialgebra*  $(\mathcal{B}, +, m, \Delta)$ , if there exists a coproduct map  $\Delta$ , that is an algebra homomorphism. Then  $(\mathcal{B}, \Delta; R)$  is a  $R$ -coalgebra.

REMARK 2.7.4 (UNIQUENESS). Usually a bialgebra is constructed by giving an algebra a coalgebra structure. However a coalgebra structure on an algebra is seldom unique. For any noncocommutative coproduct  $\Delta$  on an algebra, the twisted coproduct  $\Delta' := \sigma \circ \Delta$  affords another coproduct on the algebra. These two coproducts lead to two different bialgebras.

Most bialgebras, that I shall consider, will have a unity  $\mu$  and a counit  $\epsilon$  homomorphism and an inverse map  $S$  (anti-automorphism), which should be likened to the inverse map of a group  $G$ , that maps  $g \mapsto g^{-1}$  ( $g \in G$ ).

DEFINITION 2.7.5 (ANTIPODE). Let  $(B, +, R, m, \Delta, \mu, \epsilon)$  be a bialgebra with unity  $\mu$  and counit  $\epsilon$  homomorphisms. An algebra anti-automorphism  $S : B \rightarrow B$  which satisfies

$$\begin{aligned} S \circ m(x \otimes y) &= m \circ (S(y) \otimes S(x)) \quad x, y \in B \\ m \circ (S \times \text{id}) \circ \Delta(a) &= \mu \circ \epsilon(a) = m \circ (\text{id} \times S) \circ \Delta(a) \end{aligned}$$

is called an *antipode* map.

REMARK 2.7.6. In contrast with the bialgebra structure of an algebra, which in general is not unique, the antipode map of a bialgebra is unique.

Finally I am in a position to be able to make the following

DEFINITION 2.7.7 (HOPF ALGEBRA). A *Hopf algebra*  $(\mathcal{H}, +, R, m, \Delta, \mu, \epsilon, S)$  is a bialgebra with a unity  $(\mu : R \rightarrow \mathcal{H})$ , counit  $(\epsilon : \mathcal{H} \rightarrow R)$  and an antipode anti-automorphism  $S$ .

LEMMA 2.7.8. Let  $\mathfrak{g}$  be a  $k$ -Lie algebra and  $\mathcal{U}(\mathfrak{g})$  its universal enveloping algebra. Then  $\mathcal{U}(\mathfrak{g})$  has the structure of a Hopf algebra by extending the maps

$$\begin{aligned} \Delta(x) &:= x \otimes \mathbf{1} + \mathbf{1} \otimes x, & \Delta(a \cdot \mathbf{1}) &:= a \cdot \mathbf{1} \otimes \mathbf{1}, \\ \mu(a) &:= a\mathbf{1}, \\ \epsilon(x) &:= 0, & \epsilon(a \cdot \mathbf{1}) &:= a, \\ S(x) &:= -x, & S(\mathbf{1}) &:= 1, \end{aligned} \quad (\forall x \in \mathfrak{g}, a \in k)$$

homomorphically to all of  $\mathcal{U}(\mathfrak{g})$ .

PROOF. It is necessary to show that all the Hopf algebra maps are algebra homomorphisms. For the coproduct

$$\begin{aligned} \Delta(x \cdot y - y \cdot x) &= \Delta(x) \cdot \Delta(y) - \Delta(y) \cdot \Delta(x) \\ &= (x \cdot y) \otimes \mathbf{1} + x \otimes y + y \otimes x + \mathbf{1} \otimes (x \cdot y) \\ &\quad - (y \cdot x) \otimes \mathbf{1} - y \otimes x - x \otimes y - \mathbf{1} \otimes (y \cdot x) \\ &= (x \cdot y - y \cdot x) \otimes \mathbf{1} + \mathbf{1} \otimes (x \cdot y - y \cdot x) \\ &= \Delta([x, y]) \end{aligned}$$

□

## 2.8. Representations of algebras and Hopf algebras

One reason for the importance of Hopf algebras is that the Hopf structure of an algebra facilitates the construction of representations of the algebra, as I now explain.

A trivial representation of an algebra is one dimensional.

LEMMA 2.8.1 (TRIVIAL REPRESENTATION). Let  $\mathcal{A}$  be a  $k$ -algebra (bialgebra) with a counit homomorphism map  $\epsilon : \mathcal{A} \rightarrow k$ . Then the counit map  $\epsilon$  gives a trivial one dimensional representation of  $\mathcal{A}$  on  $k$ .



PROPOSITION 2.8.2 (TENSOR PRODUCT). *Let  $\mathcal{A}$  be a  $k$ -algebra with a bialgebra structure  $(\mathcal{A}, +, \cdot, \Delta; k)$ . Let  $(\pi_1, V_1)$  and  $(\pi_2, V_2)$  be representations of  $\mathcal{A}$ . Then  $(\pi_{12}, V_1 \otimes V_2)$ , with*

$$\pi_{12} := (\pi_1 \otimes \pi_2) \circ \Delta,$$

*is an tensor product representation of the two representations.*

The antipode map allows the construction of dual representations.

LEMMA 2.8.3 (DUAL REPRESENTATIONS). *Let  $(\mathcal{A}, \cdot; k)$  be a  $k$ -algebra, with a Hopf algebra structure  $(\mathcal{A}, +, \cdot, \Delta, S; k)$ . Let  $\omega$  be an algebra anti-automorphism of the algebra  $\mathcal{A}$ . Let  $(\pi, V)$  be a representation of the algebra  $\mathcal{A}$  and let  $V^*$  be the dual vector space of  $V$ , with the dual pairing  $\langle \cdot, \cdot \rangle$ . Defining*

$$\langle \pi^\omega(x) \cdot v^*, v \rangle := \langle v^*, \pi(\omega(x)) \cdot v \rangle,$$

*then  $(\pi^\omega, V^*)$  is the dual representation (dual  $\mathcal{A}$ -module) to  $(\pi, V)$  with respect to  $\omega$ . Note that the dual representation  $\mathcal{A} \rightarrow \text{End}(V^*)$  is also a left representation:  $\pi^*(x \cdot y) = \pi^*(x) \cdot \pi^*(y)$ .*

In particular choosing  $\omega = S$ , the dual representation  $(\pi^S, V^*)$  is obtained. If the inverse of the antipode  $S^{-1}$  exists,  $\omega = S^{-1}$  leads to another dual representation  $(\pi^{S^{-1}}, V^*)$ .

DEFINITION 2.8.4 (COREPRESENTATIONS). Let  $(\mathcal{C}, \Delta)$  be a  $k$ -coalgebra and let  $V$  be a  $k$ -vector space. A *right corepresentation* of  $(\mathcal{C}, \Delta)$  on  $V$  is a map  $\Delta_R : V \rightarrow V \otimes \mathcal{C}$  such that

$$(\Delta_R \times \text{id}) \circ \Delta_R(v) = (\text{id} \times \Delta) \circ \Delta_R(v) \quad (v \in V).$$

If further  $\mathcal{C}$  is a coalgebra with counit, then a corepresentation  $\Delta_R$  of  $(\mathcal{C}, \Delta, \epsilon)$  is required to satisfy

$$(\text{id} \times \epsilon) \circ \Delta_R(x) = \text{id}(x) \quad (x \in \mathcal{C}).$$

EXAMPLE 2.8.5. Let  $(\mathcal{A}, m)$  be an  $k$ -algebra and let  $(\mathcal{A}^*, \Delta)$  be the dual coalgebra. Let  $(\pi, V)$  be a representation of  $\mathcal{A}$  and  $V^*$  the linear vector space dual to  $V$ , then the map  $\Delta_L$

$$\langle \Delta_L(v^*), x \otimes v \rangle_{\mathcal{A} \otimes V} := \langle v^*, \pi(x) \cdot v \rangle$$

is a left corepresentation of  $\mathcal{A}^*$ .

## 2.9. Yang-Baxter equations and $\mathcal{R}$ -matrices

When considering the tensor product representations of a (bi)algebra, an interesting problem is determining under what conditions  $(\pi_{12}, V_1 \otimes V_2)$  and  $(\pi_{21}, V_2 \otimes V_1)$  are equivalent. It is perhaps surprising that for a general Hopf algebra it is a nontrivial problem to find an intertwiner between these two tensor product representations.

DEFINITION 2.9.1 ( $\mathcal{R}$ -MATRIX). Let  $(\mathcal{A}, \cdot, \Delta; k)$  be a  $k$ -bialgebra. Let  $(\pi_1, V_1)$  and  $(\pi_2, V_2)$  be representations of the  $k$ -algebra  $(\mathcal{A}, \cdot)$ . If the tensor product representations  $\pi_{12} := (\pi_1 \times \pi_2) \circ \Delta$  and  $\pi_{21} := (\pi_2 \times \pi_1) \circ \Delta$  are equivalent, then there exists an intertwiner  $\mathcal{R}_{12} : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$ :

$$\mathcal{R}_{12} \circ (\pi_1 \otimes \pi_2) \circ \Delta(x) = (\pi_2 \otimes \pi_1) \circ \mathcal{R}_{12} \circ \Delta(x) \quad x \in \mathcal{A}.$$

LEMMA 2.9.2. Let  $(\mathcal{A}, \cdot, \Delta; k)$  be a  $k$ -bialgebra and  $(\pi_i, V_i)$  ( $i \in \{1, 2, 3\}$ ) be representations of the  $k$ -algebra  $(\mathcal{A}, \cdot)$ .

If the following conditions are satisfied

- (1) there exist intertwiners  $\mathcal{R}_{ij} : V_i \otimes V_j \rightarrow V_j \otimes V_i$  (for all pairs  $(i, j)$  such that  $i \neq j$ ) from the tensor product representation  $(\pi_{ij}, V_i \otimes V_j)$  to  $(\pi_{ji}, V_j \otimes V_i)$  and
- (2) the representations  $(\pi_{ijk}, V_i \otimes V_j \otimes V_k)$  ( $\pi_{ijk} := (\pi_i \times \pi_j \times \pi_k) \circ (\Delta \times \text{id}) \circ \Delta : \mathcal{A} \rightarrow \text{End}(V_i) \otimes \text{End}(V_j) \otimes \text{End}(V_k)$ ) are irreducible,

then two intertwiners from the representation  $(\pi_{123}, V_1 \otimes V_2 \otimes V_3)$  to  $(\pi_{321}, V_3 \otimes V_2 \otimes V_1)$  can be constructed from the  $R$ -matrices  $\mathcal{R}_{ij}$  and their equivalence (up to a scalar  $\rho \in k$ ) is expressed as

$$(\mathcal{R}_{23} \times \text{id}) \circ (\text{id} \times \mathcal{R}_{13}) \circ (\mathcal{R}_{12} \times \text{id}) = \rho(\text{id} \times \mathcal{R}_{12}) \circ (\mathcal{R}_{13} \times \text{id}) \circ (\text{id} \times \mathcal{R}_{23}).$$

This is called the *quantum Yang-Baxter Equation without spectral parameter*. The lemma is proved for example in [Jim92, 2.1].

## Part I

### Finite Quantum Groups



## CHAPTER 3

### Matrix Quantum Groups and Quantum Spaces

In classical differential geometry it is the commutative  $k$ -algebra  $C^\infty(M, k)$  of smooth  $k$ -valued functions on a smooth manifold  $M$  that is of central importance.  $C^\infty(M, k)$  contains in particular the functions, whose restriction to local open sets in  $M$ , gives  $M$  a local coordinate structure. In noncommutative differential geometry, noncommutative algebras take the role that  $C^\infty(M, k)$  had classically.

#### 3.1. The quantum groups of $GL_2$ and $SL_2$

I start with two simple examples of matrix quantum groups: the quantum groups corresponding to the Lie groups  $GL_2$  and  $SL_2$ .

**3.1.1.** Recall that the Lie group  $GL_2(\mathbb{C})$  (from here on denoted  $GL_2$ ) of complex  $2 \times 2$  matrices with nonzero determinant can also be defined in terms of its natural coordinates  $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}$ , whose values on a matrix  $g \in GL_2$  are the entries in the matrix:

$$g = \begin{pmatrix} \bar{a}(g) & \bar{b}(g) \\ \bar{c}(g) & \bar{d}(g) \end{pmatrix}.$$

Of course the algebra  $\text{fun}(GL_2, \mathbb{C})$  of polynomial functions (in  $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}$ ) is commutative: the coordinate functions commute. They also satisfy the relation  $\bar{a}(g)\bar{d}(g) - \bar{b}(g)\bar{c}(g) \neq 0$  for all  $g \in GL_2$ .

The Lie group  $SL_2$  is defined to be the group formed by the subset  $\{g \in GL_2 \mid \det(g) = 1\}$  of  $GL_2$ , the elements of  $GL_2$  with unit determinant. Since  $SL_2$  is a subgroup of  $GL_2$ , I define  $\text{fun}(SL_2, \mathbb{C})$  to be the restriction of  $\text{fun}(GL_2, \mathbb{C})$  to  $SL_2$ :  $\text{fun}(SL_2, \mathbb{C}) := \text{fun}(GL_2, \mathbb{C})|_{SL_2}$ . I use the same symbols  $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}$  to denote the commutative coordinate functions of  $GL_2$  when restricted to  $SL_2$ . They satisfy the relation  $\bar{a}(g)\bar{d}(g) - \bar{b}(g)\bar{c}(g) = 1$  for all  $g \in SL_2$ . From here on, write  $\text{fun}(GL_2)$  and  $\text{fun}(SL_2)$  respectively for  $\text{fun}(GL_2, \mathbb{C})$  and  $\text{fun}(SL_2, \mathbb{C})$ .

**NOTATION.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be associative  $k$ -algebra. Consider then two  $n \times n$  matrices  $U = (u_{ij})_{i,j \in [1,n]}$  and  $V = (v_{ij})_{i,j \in [1,n]}$ , with elements  $u_{ij} \in \mathcal{A}, v_{ij} \in \mathcal{B}$ . Then I define the *matrix tensor product*  $U \dot{\otimes} V$  of  $U$  and  $V$  to be the  $n \times n$  matrix with entries

$$(U \dot{\otimes} V)_{ik} := \sum_{j=1}^n u_{ij} \otimes v_{jk} \quad \in \mathcal{A} \otimes \mathcal{B}.$$

PROPOSITION 3.1.2. *The polynomial function algebra  $\text{fun}(SL_2)$  is a commutative, noncocommutative Hopf algebra.*

*The coproduct  $\Delta : \text{fun}(SL_2) \rightarrow \text{fun}(SL_2) \otimes \text{fun}(SL_2)$  is given by*

$$\Delta : \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \dot{\otimes} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}, \quad \Delta : \mathbf{1} \mapsto \mathbf{1} \otimes \mathbf{1}.$$

*The antipode map  $S : \text{fun}(SL_2) \rightarrow \text{fun}(SL_2)$  is*

$$S : \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \mapsto \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix}, \quad S : \mathbf{1} \mapsto \mathbf{1} \in \text{fun}(SL_2).$$

*The counit  $\epsilon : \text{fun}(SL_2) \rightarrow \mathbb{C}[q, q^{-1}]$  is*

$$\epsilon : \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \epsilon : \mathbf{1} \mapsto 1 \in \mathbb{C}.$$

*The coproduct and counit extend homomorphically and the antipode antihomomorphically to the whole of  $\text{fun}(SL_2)$ .*

REMARK 3.1.3. In order to extend this Hopf algebra structure to  $\text{fun}(GL_2)$  it is necessary to ‘complete’ it by adding an element  $\bar{D}^{-1}$  to it.  $\bar{D}^{-1}$  is the essentially the reciprocal of the determinant function of  $\text{fun}(GL_2)$ , which cannot be expressed as a polynomial of the coordinate functions  $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}$ .

DEFINITION 3.1.4 ( $\text{fun}_q(GL_2)$ ). Let  $q$  be an indeterminant with inverse  $q^{-1}$ . The quantum matrix group (or quantised polynomial function algebra)  $\text{fun}_q(GL_2)$  is defined to be the associative unital  $\mathbb{C}[q, q^{-1}]$ -algebra with generators  $\{a, b, c, d, D^{-1}\}$  which satisfy the relations

$D^{-1}$  is central,

$$a \cdot b = q b \cdot a,$$

$$a \cdot c = q c \cdot a,$$

$$b \cdot d = q d \cdot b,$$

$$c \cdot d = q d \cdot c,$$

$$b \cdot c = c \cdot b,$$

$$a \cdot d - d \cdot a = (q - q^{-1}) b \cdot c,$$

$$D^{-1} \cdot (ad - qbc) = 1,$$

$$(ad - qbc) \cdot D^{-1} = 1.$$

So  $\text{fun}_q(GL_2) := \mathbb{C}[q, q^{-1}][[a, b, c, d, D^{-1}]] / \mathcal{I}_q$ , where  $\mathcal{I}_q$  is the two-sided ideal generated by the above relations.

REMARK 3.1.5.  $\text{fun}_q(GL_2) = \sum_{k,l,m,n,p \in \mathbb{N}} \mathbb{C}[q, q^{-1}] a^k b^l c^m d^n D^{-n}$  as a  $\mathbb{C}[q, q^{-1}]$ -module (though this basis is overcomplete).

**3.1.6.** It is often useful to do calculations in  $\text{fun}_q(GL_2)$  using the following matrix of the generators  $\{a, b, c, d\}$

$$T := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

**LEMMA 3.1.7.** *The element  $\det_q(T) := ad - qbc \equiv da - q^{-1}bc$  is central in  $\text{fun}_q(GL_2)$ . It is called the quantum determinant of  $\text{fun}_q(GL_2)$ .*

The generator  $D^{-1}$  can be thought of as the inverse of the quantum determinant.

**PROPOSITION 3.1.8.** *The quantum group function algebra  $\text{fun}_q(GL_2)$  has the structure of a noncommutative, noncocommutative Hopf algebra.*

*The coproduct  $\Delta : \text{fun}_q(GL_2) \rightarrow \text{fun}_q(GL_2) \otimes \text{fun}_q(GL_2)$  is given by*

$$\Delta : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \Delta : D^{-1} \mapsto D^{-1} \otimes D^{-1}.$$

*The antipode map  $S : \text{fun}_q(GL_2) \rightarrow \text{fun}_q(GL_2)$  is*

$$S : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto D^{-1} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}, \quad S : D^{-1} \mapsto ad - qbc.$$

*The counit  $\epsilon : \text{fun}_q(GL_2) \rightarrow \mathbb{C}[q, q^{-1}]$  is*

$$\epsilon : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \epsilon : D^{-1} \mapsto 1.$$

*The coproduct and counit extend homomorphically and the antipode antihomomorphically to the whole of  $\text{fun}_q(GL_2)$ . The Hopf algebra maps act on the unity as  $\Delta(1) = 1 \otimes 1$ ,  $\epsilon(1) = 1 \in \mathbb{C}[q, q^{-1}]$  and  $S(1) = 1 \in \text{fun}_q(GL_2)$ .*

*Note also the matrix products:  $T \cdot S(T) = 1$  and  $S(T) \cdot T = 1$ .*

**PROOF.** The proposition is easily proved by checking that all the maps are consistent and satisfy the relations required of a Hopf algebra.  $\square$

**3.1.9. The quantum group of  $SL_2$ .** [FT86] The quantum group  $\text{fun}_q(SL_2)$  is defined to be the quotient algebra of  $\text{fun}_q(GL_2)$  generated by only  $\{a, b, c, d\}$  with unit quantum determinant ( $\det_q(T) = 1$ ) satisfying the relations of  $\text{fun}_q(GL_2)$  (see 3.1.4). So

$$\text{fun}_q(SL_2) := \text{fun}_q(GL_2) / \langle D^{-1} - 1, ad - qbc - 1 \rangle.$$

So the quantum group  $\text{fun}_q(SL_2)$  is a quotient algebra of the quantum group  $\text{fun}_q(GL_2)$ .

**PROPOSITION 3.1.10.** *Let  $\{a, b, c, d\}$  and  $\{a', b', c', d'\}$  be two sets of noncommutative coordinates of  $\text{fun}_q(SL_2)$ , both satisfying the relations of 3.1.4. Let*

$$T := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad T' := \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Consider the maps  $L, R : \text{fun}_q(SL_2) \rightarrow \text{fun}_q(SL_2) \otimes \text{fun}_q(SL_2)$  given by the matrix tensor products  $L : T \mapsto T \dot{\otimes} T'$  and  $R : T \mapsto T' \dot{\otimes} T$

$$\begin{aligned} L : \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \\ R : \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

The maps  $L$  and  $R$  are  $\mathbb{C}[q, q^{-1}]$ -algebra homomorphisms.

A special case of this result, is the coproduct map  $\Delta : T \mapsto T \dot{\otimes} T$ , which is a homomorphism  $\text{fun}_q(SL_2) \rightarrow \text{fun}_q(SL_2) \otimes \text{fun}_q(SL_2)$ .

**3.1.11. Classical limit.** Let  $\epsilon \in \mathbb{C}^\times$ . I define the specialisation  $\text{fun}_\epsilon(SL_2)$  of  $\text{fun}_q(SL_2)$  at  $q = \epsilon$ , to be  $\text{fun}_\epsilon(SL_2) := \text{fun}_q(SL_2) / \langle q - \epsilon \rangle$ .

A particularly important specialisation of  $\text{fun}_q(SL_2)$  is  $\text{fun}_1(SL_2)$  at  $q = 1$ .

**LEMMA 3.1.12.** *The  $\mathbb{C}$ -algebra  $\text{fun}_1(SL_2)$  is a commutative algebra. It is Hopf algebra isomorphic to  $\text{fun}(SL_2)$ .*

**PROOF.** It is clear from the definition of  $\text{fun}_\epsilon(SL_2)$  that at  $\epsilon = 1$ ,  $\text{fun}_\epsilon(SL_2)$  becomes a commutative algebra. Further  $ad - bc = 1$ . This proves the algebra isomorphism  $\text{fun}_1(SL_2) \simeq \text{fun}(SL_2)$ . The Hopf algebra isomorphism follows similarly.  $\square$

### 3.2. The Quantum Plane

Lie groups occur often as the transformation or symmetry groups of a vector space (perhaps with certain additional structures (for example inner products or a symplectic structure), which may also be invariant under the group action). It is interesting to consider what the analogue of an action on a linear space is for a matrix quantum group.

**3.2.1.** Consider an  $n$ -dimensional  $\mathbb{C}$ -vector space  $V$ . Let  $V^*$  be the vector space dual to  $V$ . A (dual) basis in  $V^*$  forms a set of linear coordinates on  $V$ . Let  $\{\bar{x}_i \mid i = 1, \dots, n\}$  be a basis of  $V^*$ . The  $\mathbb{C}$ -span of all monomials in these coordinates has a natural structure of a commutative algebra  $\text{fun}(V, \mathbb{C})$  ( $\text{fun}(V)$ ) (the polynomial function algebra on  $V$ ) and is isomorphic to the symmetric algebra  $S(V^*)$  over  $V^*$ .

**DEFINITION 3.2.2.** [Man89] Let  $q$  be an indeterminant with inverse  $q^{-1}$ . The *quantum vector space* (or quantum function space)  $\text{fun}_q(V)$  of dimension  $n$  is an associative  $\mathbb{C}[q, q^{-1}]$ -algebra with unity and  $n$  generators  $\{x_1, \dots, x_n\}$  which satisfy the relations:

$$x_i \cdot x_j = qx_j \cdot x_i \quad (1 \leq i < j \leq n).$$

Then  $\text{fun}_q(V) := \mathbb{C}[q, q^{-1}][[x_1, \dots, x_n]] / \mathcal{J}_q$ , where  $\mathcal{J}_q$  is the two-sided ideal generated by the above relations.



**3.3. Coaction of  $\text{fun}_q(SL_2)$  on the quantum plane  $\text{fun}_q(V_2)$** 

Next I consider how a quantum group function algebra co-acts on a quantum vector space. Since the objects are defined in terms of the dual (noncommutative) function algebras, the arrows in the maps are reversed relative to the usual maps of a (Lie) group acting on a vector space.

**DEFINITION 3.3.1 (COACTION).** Let  $k$  be a field. A left coaction of a  $k$ -bialgebra  $\mathcal{B}$  (with counit  $\epsilon$ ) on a  $k$ -algebra  $\mathcal{V}$ , is a  $k$ -algebra homomorphism  $\Delta_L : \mathcal{V} \rightarrow \mathcal{B} \otimes \mathcal{V}$ , which satisfies

$$\begin{aligned} (\text{id} \times \Delta_L) \circ \Delta_L &= (\Delta \times \text{id}) \circ \Delta_L & (\mathcal{V} \mapsto \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{V}) \\ (\epsilon \times \text{id}) \circ \Delta_L &= i_k & (\mathcal{V} \rightarrow k \otimes \mathcal{V}). \end{aligned}$$

Here  $i_k$  is the isomorphism  $\mathcal{V} \xrightarrow{\simeq} k \otimes \mathcal{V}$ . When the bialgebra  $\mathcal{B}$  co-acts on  $\mathcal{V}$ , then  $\mathcal{V}$  is called a left  $\mathcal{B}$ -comodule (or a left corepresentation) of  $\mathcal{B}$ . A right coaction ( $\mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{B}$ ) and right comodule are similarly defined (c.f. 2.8.4).

Consider now a 2 dimensional  $\mathbb{C}$ -vector space  $V_2$  and the quantum vector space  $\text{fun}_q(V_2)$ . Call  $\text{fun}_q(V_2)$  the quantum plane.

**LEMMA 3.3.2.** Write  $\{x, y\}$  ( $xy = qyx$ ) for the generators  $\{x_1, x_2\}$  of the quantum plane  $\text{fun}_q(V_2)$ . The quantum plane  $\text{fun}_q(V_2)$  is a  $\text{fun}_q(SL_2)$ -comodule. The left coaction  $\Delta_L : \text{fun}_q(V_2) \rightarrow \text{fun}_q(SL_2) \otimes \text{fun}_q(V_2)$  is given by

$$\Delta_L : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} x \\ y \end{pmatrix}.$$

**PROOF.** I check that  $\Delta_L(x \cdot y) = q\Delta_L(y \cdot x)$ .

$$\begin{aligned} \Delta_L(xy) &= \Delta_L(x) \cdot \Delta_L(y) \\ &= (a \otimes x + b \otimes y) \cdot (c \otimes x + d \otimes y) \\ &= ac \otimes xx + ad \otimes xy + bc \otimes yx + bd \otimes yy \\ &= q(ca \otimes xx) + q(da + (q - q^{-1})bc) \otimes yx + bc \otimes yx + q(db \otimes yy) \\ &= q(ca \otimes xx + da \otimes yx + bc \otimes xy + db \otimes yy) \\ &= q(c \otimes x + d \otimes y) \cdot (a \otimes x + b \otimes y) \\ &= q\Delta_L(y) \cdot \Delta_L(x). \end{aligned}$$

□

**3.3.3.** Let  $V$  be a  $\mathbb{C}$ -vector space and let  $\epsilon \in \mathbb{C}^\times$ . Define the specialisation  $\text{fun}_\epsilon(V)$  of  $\text{fun}_q(V)$  at  $q = \epsilon$  to be the  $\mathbb{C}$ -algebra  $\text{fun}_\epsilon(V) := \text{fun}_q(V) / \langle q - \epsilon \rangle$ .

**LEMMA 3.3.4.**  $\text{fun}_1(V) \simeq \text{fun}(V)$  as  $\mathbb{C}$ -algebras.

### 3.4. Representations of Quantum groups

NOTATION. In a bialgebra, let the notation

$$\Delta(x) = \sum_i x_{(1,i)} \otimes x_{(2,i)},$$

denote the components of the coproduct.

Recall that in a Lie group  $G$  there are some natural actions of the group on itself. There is the left action  $l_g : g' \mapsto gg'$  and the right action  $r_g : g' \mapsto g'g$  and the adjoint action  $\text{Ad}_g : g' \mapsto gg'g^{-1}$ . The adjoint map  $\text{Ad}_g$  acts as an automorphism of  $G$  for each  $g \in G$ . For a quantum group the equivalent of the left and right actions is essentially given by the coproduct (see 3.1.10).

DEFINITION 3.4.1 (ADJOINT ACTION). The *quantum adjoint action* of a Hopf algebra  $\mathcal{H}$  on itself is defined by

$$\begin{aligned} \text{Ad}(x) : \mathcal{H} &\rightarrow \mathcal{H}, \\ \text{Ad}(x) : y &\mapsto \sum_i x_{(1,i)} y S(x_{(2,i)}), \quad (x, y \in \mathcal{H}). \end{aligned}$$

The map  $\text{Ad}(x) : \text{fun}_q(SL_2) \rightarrow \text{fun}_q(SL_2)$  ( $x \in \text{fun}_q(SL_2)$ ) gives the adjoint representation of  $\text{fun}_q(SL_2)$  on itself.

### 3.5. $\text{fun}_\epsilon(SL_2)$ and $\text{fun}_\epsilon(V_2)$ at a root of unity

PROPOSITION 3.5.1. Let  $\ell$  be an integer such that  $\ell > 2$ . Let  $\epsilon = e^{\frac{2\pi i}{\ell}}$  (a primitive  $\ell$ -th root of unity).

- (a) In  $\text{fun}_\epsilon(V)$ , the elements  $\{x_i^\ell \mid i \in [1, n]\}$  are central.
- (b) In  $\text{fun}_\epsilon(SL_2)$ , the elements  $\{a^\ell, b^\ell, c^\ell, d^\ell\}$  are central.

PROOF. (a) In  $\text{fun}_\epsilon(V_2)$ ,  $x^\ell y = \epsilon^\ell y x^\ell = y x^\ell$ , so  $x^\ell$  is central. Similarly  $y^\ell$  is central. (b) That  $b^\ell$  and  $c^\ell$  are central, follows from (a). To show that  $a^\ell$  is central it is necessary to check only that  $a^\ell d = d a^\ell$ , since it clearly commutes with  $b$  and  $c$ . Using the relation  $ad - da = (q - q^{-1})bc$ , the following identity is proved by induction

$$a^\ell \cdot d = d \cdot a^\ell + (\epsilon - \epsilon^{-1})\epsilon^{\ell-1} [\ell]_\epsilon b \cdot c \cdot a^{\ell-1}.$$

Since  $[\ell]_\epsilon = 0$ ,  $a^\ell$  is central in  $\text{fun}_\epsilon(SL_2)$ . The proof that  $d^\ell$  is central is similar.  $\square$

COROLLARY 3.5.2. (a) Let  $\mathcal{Z}_V$  be the central subalgebra of  $\text{fun}_\epsilon(V)$  generated by  $\{x_i^\ell \mid i \in [1, n]\}$ .  $\text{fun}_\epsilon(V)$  is finite dimensional over  $\mathcal{Z}_V$ , i.e.  $\text{fun}_\epsilon(V)$  is a free module over  $\mathcal{Z}_V$  with basis  $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$  ( $m_i \in [0, \ell - 1]$ ,  $i \in [1, n]$ ).

(b) Let  $\mathcal{Z}_{SL_2}$  be the central subalgebra of  $\text{fun}_\epsilon(SL_2)$  generated by  $\{a^\ell, b^\ell, c^\ell, d^\ell\}$ . Then  $\text{fun}_\epsilon(SL_2)$  is finite dimensional over  $\mathcal{Z}_{SL_2}$ :  $\text{fun}_\epsilon(SL_2)$  is a free module over  $\mathcal{Z}_{SL_2}$  with basis  $a^{m_1} b^{m_2} c^{m_3} d^{m_4}$  ( $m_1, m_2, m_3, m_4 \in [0, \ell - 1]$ ).

REMARK 3.5.3. Representations of  $\text{fun}_\epsilon(SL_2)$  at a root of unity have been studied in [KP94]. The case of a general quantum matrix group at a root of unity is described in [DCL93].

LEMMA 3.5.4. *The map  $\omega : \text{fun}_q(SL_2) \rightarrow \text{fun}_q(SL_2)$  given by*

$$\omega : T \mapsto S(T)^t, \quad \omega : q \mapsto q,$$

*is an anti-automorphism of  $SL_2$ .*

PROOF. It is easily checked that  $\omega$  is an anti-automorphism of  $\text{fun}_q(SL_2)$ . Here I check that

$$\begin{aligned} \omega(ab) - q\omega(ba) &= \omega(b)\omega(a) - q\omega(a)\omega(b) \\ &= -q^{-1}(cd - qdc) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \omega(ad - da - (q - q^{-1})bc) &= ad - da - (q - q^{-1})bc \\ &= 0. \end{aligned}$$

The rest of the relations of  $\text{fun}_q(SL_2)$  are similarly shown to map into each other under  $\omega$ .  $\square$

### 3.6. The 2-parameter Quantum Group $\text{fun}_{pq}(GL_2)$

**3.6.1.** Let  $p, q$  be indeterminants with inverses  $p^{-1}, q^{-1}$ . Let  $\mathbb{C}[p, p^{-1}, q, q^{-1}]$  be the ring of polynomials in  $p, q$  and their inverses. Consider the associative  $\mathbb{C}[p, p^{-1}, q, q^{-1}]$ -algebra  $\text{fun}_{pq}(GL_2)$  generated by  $\{a, b, c, d, D^{-1}\}$  satisfying the relations

$$\begin{aligned} a \cdot b &= p b \cdot a, & a \cdot c &= q c \cdot a, \\ b \cdot d &= q d \cdot b, & c \cdot d &= p d \cdot c, \\ p b \cdot c &= q c \cdot b, & a \cdot d - d \cdot a &= (p - q^{-1}) b \cdot c, \\ D^{-1} \cdot a &= a \cdot D^{-1}, & p D^{-1} \cdot b &= q b \cdot D^{-1}, \\ q D^{-1} \cdot c &= p c \cdot D^{-1}, & D^{-1} \cdot d &= d \cdot D^{-1}, \\ D^{-1} \cdot (a \cdot d - q c \cdot b) &= 1. \end{aligned}$$

This is a 2-parameter deformation of  $\text{fun}(GL_2)$ . Note that  $ad - qcb$  is not central in  $\text{fun}_{pq}(GL_2)$ , so there is no simple way of defining a 2-parameter deformation of  $\text{fun}(SL_2)$  as a quotient of this algebra.

**3.6.2.**  $\text{fun}_{pq}(GL_2)$  is a Hopf algebra. The coproduct  $\Delta : \text{fun}_{pq}(GL_2) \rightarrow \text{fun}_{pq}(GL_2) \otimes \text{fun}_{pq}(GL_2)$  is given by

$$\Delta : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \Delta : D^{-1} \mapsto D^{-1} \otimes D^{-1}.$$

The antipode map  $S : \text{fun}_{pq}(GL_2) \rightarrow \text{fun}_{pq}(GL_2)$  is

$$S : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto D^{-1} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}, \quad S : D^{-1} \mapsto ad - pbc.$$

The counit  $\epsilon : \text{fun}_{pq}(GL_2) \rightarrow \mathbb{C}[q, q^{-1}]$  is

$$\epsilon : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \epsilon : D^{-1} \mapsto 1.$$

The Hopf algebra maps act on the unity as  $\Delta(1) = 1 \otimes 1$ ,  $\epsilon(1) = 1 \in \mathbb{C}[p, p^{-1}, q, q^{-1}]$  and  $S(1) = 1 \in \text{fun}_{pq}(GL_2)$ .

REMARK 3.6.3. There is a Hopf algebra isomorphism  $\text{fun}_{pq}(GL_2)/\langle p-q \rangle \simeq \text{fun}_q(GL_2)$ .

### 3.7. Quantum phase space

The algebra  $Q_{\hbar} := \mathbb{C}[[x, p]] / \langle xp - px - \hbar i \rangle$  can heuristically be thought of as the general ‘quantum mechanical phase space of a particle in one dimension’ (with ‘non-commutative phase space coordinates’  $x$  and  $p$ ) and  $Q_{q, \hbar} := \mathbb{C}[[x, p]] / \langle xp - qpx - \hbar i \rangle$  as a ‘deformed quantum phase space’. 1-parameter deformations of bosonic and fermionic quantum mechanical phase space and their symmetries have been studied by Zumino [Zum91] in the  $\mathcal{R}$ -matrix formalism.

## CHAPTER 4

# Quantum enveloping algebras

### 4.1. Introduction

In 1985 Jimbo and Drinfeld independently introduced [Dri85, Jim85] a  $q$ -analogue of the universal enveloping algebra of every simple (and affine) Lie algebra. In this chapter I describe some results, due mainly to Lusztig and to De Concini and Kac, on the quantum enveloping algebra of a simple (finite) Lie algebra and its ‘specialisation’ at an odd primitive root of unity. The techniques and results described here will be assumed in chapters 7 and 8, when I come to consider quantum affine (Kac-Moody) algebras and their specialisation at a root of unity. Let me summarise the ideas and contents of this chapter.

**4.1.1.** In the spirit of Kac, to every symmetrisable, positive definite Cartan matrix  $(a_{ij})$ , there exists an associated unique simple finite dimensional Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$  and its universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$ . Also associated to  $(a_{ij})$ , there is a set of fundamental weights, simple roots, a weight lattice, a root lattice, a braid group, a Weyl group and so on. The Weyl group acts naturally on the root lattice and induces a corresponding automorphic action of the braid group on  $\mathfrak{g}$  and  $\mathcal{U}(\mathfrak{g})$ . In particular the Weyl group action on the simple roots, which generates the complete root system of  $\mathfrak{g}$ , lifts to an action of the braid group on the Chevalley generators. This action generates all the root vectors of  $\mathfrak{g}$ . To each reduced expression of the longest element of the Weyl group, there is an associated basis of the root vectors of  $\mathfrak{g}$ . Finally, as I mentioned in 2.7.8,  $\mathcal{U}(\mathfrak{g})$  is a cocommutative Hopf algebra.

In the quantum case, Drinfeld and Jimbo have associated to each  $(a_{ij})$  a quantum enveloping algebra  $\mathcal{U}_q(\mathfrak{g})$  with relations in terms of a quantum Chevalley presentation. In this chapter I concentrate on the case when the Cartan matrix is of finite type, i.e. that of a simple finite dimensional Lie algebra  $\mathfrak{g}$ . (The quantum groups associated to affine Cartan matrices are considered in part III.) The quantum enveloping algebra  $\mathcal{U}_q(\mathfrak{g})$  is a noncocommutative Hopf algebra. At the specialisation  $q = 1$ , it can be shown rigorously that a certain subalgebra of  $\mathcal{U}_q(\mathfrak{g})$  becomes isomorphic (modulo certain central elements) to the classical universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$ . In the quantum case the problem of finding a basis of root vectors is a rather more difficult problem than for a classical Lie algebra. This is basically because the Hopf algebra is noncocommutative (or equivalently the quantum adjoint action acts essentially like a

$q$ -deformed commutator). This problem was solved by Lusztig, who proved that the Weyl group action on simple roots can also be lifted to an automorphic action of the braid group on  $\mathcal{U}_q(\mathfrak{g})$ . He further showed that, in analogy with the classical case, to each reduced expression of the longest element of the Weyl group there is an associated basis of root vectors of  $\mathcal{U}_q(\mathfrak{g})$ .

The representation theory of  $\mathcal{U}_q(\mathfrak{g})$  has been developed to a quite mature state (at least in the case of  $q$  generic). The fundamental highest weight modules are the Verma modules over  $\mathcal{U}_q(\mathfrak{g})$ . All other highest weight representations generated by a single vector can be constructed out of the Verma modules and their quotients.

**4.1.2.** A remarkable facet of quantum group theory is the case when  $q$  is a primitive  $\ell$ -th root of unity  $\epsilon$ . There exists a specialisation  $\mathcal{U}_\epsilon(\mathfrak{g})$  of  $\mathcal{U}_q(\mathfrak{g})$  at  $q = \epsilon$ . In this case the centre of  $\mathcal{U}_\epsilon(\mathfrak{g})$  is much enlarged. The large centre is most easily constructed as follows. First one proves that the Chevalley generators to the  $\ell$ -th power lie in the centre of  $\mathcal{U}_\epsilon(\mathfrak{g})$ . Then by successively applying the braid group generators to these central elements, all the new central elements are found. Lusztig, De Concini, Kac and Procesi think of the root of unity case as a  $q$ -analogue of a simple finite Lie algebra over a field of positive characteristic  $\ell$ .

With this information it is possible to study the representations of  $\mathcal{U}_\epsilon(\mathfrak{g})$  at a root of unity. Because of the large centre it turns out that every Verma module over  $\mathcal{U}_\epsilon(\mathfrak{g})$  is reducible, and has a natural reduction to a (finite dimensional) ‘diagonal module’. More interestingly at a root of unity there exist finite dimensional modules over  $\mathcal{U}_\epsilon(\mathfrak{g})$  that are not quotients of a Verma module: these include ‘cyclic’ (‘periodic’) modules and ‘semicyclic’ (‘semiperiodic’) modules, where all or some of the Chevalley generators act injectively. Unfortunately I do not have time or space to describe the parametrisation of the irreducible finite dimensional representations of  $\mathcal{U}_\epsilon(\mathfrak{g})$  at a root of unity, due to De Concini, Kac and Procesi.

The case of  $\mathcal{U}_q(\mathfrak{sl}_2)$  is the simplest. In this case it is rather straightforward to write down all the irreducible finite dimensional representation both at  $q$  generic and at  $q$  a root of unity.

## 4.2. The Lie algebra $\mathfrak{g}$

**4.2.1.** Let  $(a_{ij})_{i,j \in [1,r]}$  be an  $r \times r$  matrix with integer entries, such that the diagonal elements  $a_{ii} = 2$ , the off-diagonal elements  $a_{ij} \in \{0, -1, -2, -3\}$  ( $i \neq j$ ) and there exist  $r$  positive coprime integers  $d_i$  ( $i \in [1, r]$ ) such that  $(d_i a_{ij})$  is a symmetric positive definite matrix. Then  $(a_{ij})$  is the Cartan matrix of a simple finite Lie algebra  $\mathfrak{g}$  of rank  $r$ .

NOTATION. As usual I define in a Lie algebra the adjoint map to be  $\text{ad}(x)y := [x, y]$ .

**4.2.2.** The Lie algebra  $\mathfrak{g}$  associated to the Cartan matrix  $(a_{ij})$  has the following relations over  $\mathbb{C}$  among its Chevalley presentation generators  $\{\bar{h}_i, \bar{e}_i, \bar{f}_i \mid i \in [1, r]\}$ :

$$\begin{aligned} [\bar{h}_i, \bar{h}_j] &= 0, & [\bar{e}_i, \bar{f}_j] &= \delta_{ij} \bar{h}_i, \\ [\bar{h}_i, \bar{e}_j] &= a_{ij} \bar{e}_j, & [\bar{h}_i, \bar{f}_j] &= -a_{ij} \bar{f}_j, \\ \text{ad}(\bar{e}_i)^{1-a_{ij}}(\bar{e}_j) &= 0, & \text{ad}(\bar{f}_i)^{1-a_{ij}}(\bar{f}_j) &= 0. \end{aligned}$$

The last two relations are called the Serre-Chevalley relations (or Serre relations for short). Let  $\mathcal{U}(\mathfrak{g})$  be the universal enveloping algebra of  $\mathfrak{g}$ , as defined in 2.5.10.

### 4.3. Lattices, Braid group and Weyl group

**4.3.1. Weight lattice.** Let  $P$  be a free abelian group over  $\mathbb{Z}$  (or  $\mathbb{Z}$ -lattice) with generators  $\{\omega_i \mid i \in [1, r]\}$

$$P := \sum_{i \in [1, r]} \mathbb{Z}\omega_i.$$

$P$  is called the weight lattice of  $\mathfrak{g}$ .

Let  $Q^\vee := \text{Hom}(P, \mathbb{Z})$  be the dual lattice to  $P$  with a dual basis  $\Pi^\vee := \{\alpha_i^\vee \mid i \in [1, r]\}$ , so that under the dual pairing  $\langle \cdot, \cdot \rangle : P \times Q^\vee \rightarrow \mathbb{Z}$

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}.$$

$Q^\vee$  is called the coroot lattice of  $\mathfrak{g}$ .

**4.3.2. Root lattice.** Define the elements

$$\alpha_i := \sum_{j \in [1, r]} a_{ij} \omega_j \in P.$$

They generate a free abelian subgroup (sub-lattice) of  $P$ :

$$Q := \sum_{i \in [1, r]} \mathbb{Z}\alpha_i \subset P,$$

called the root lattice of  $\mathfrak{g}$ . The pairing on  $P \times Q^\vee$  restricts to  $Q \times Q^\vee$  as

$$\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}.$$

Let  $P_+ := \sum_{i \in [1, r]} \mathbb{N}\omega_i$  (the positive weights) and  $Q_+ := \sum_{i \in [1, r]} \mathbb{N}\alpha_i$ . Define a partial ordering  $\geq$  in  $P$  by  $\lambda \geq \mu$ , if  $\lambda - \mu \in P_+$ .

**4.3.3. Symmetric form.** Define a bilinear map  $(\cdot, \cdot) : P \times Q \rightarrow \mathbb{Z}$  by

$$(\omega_i, \alpha_j) := d_i \delta_{ij}.$$

Note that the restriction of this map to  $Q \times Q$  defines a symmetric  $\mathbb{Z}$ -valued bilinear form on  $Q$ , since

$$(\alpha_i, \alpha_j) = d_i a_{ij}.$$

**4.3.4.** Define the following numbers from the Cartan matrix

$$m_{ij} := \frac{\pi}{\arccos(\frac{\sqrt{(a_{ij}a_{ji})}}{2})} \quad (i \neq j).$$

Then  $m_{ij} = 2, 3, 4, 6$  ( $i \neq j$ ), when  $a_{ij}a_{ji} = 0, 1, 2, 3$  respectively. Note that  $m_{ij} = m_{ji}$ .

**DEFINITION 4.3.5 (BRAID GROUP).** The Braid group  $B$  associated to the Cartan matrix  $(a_{ij})$  is generated by  $\{T_i, T_i^{-1}, 1 \mid i \in [1, r]\}$ . They satisfy the relations

$$\begin{aligned} T_i T_i^{-1} &= 1 = T_i^{-1} T_i, \\ \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} &= \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}} \quad (i \neq j). \end{aligned}$$

The Braid group  $B$  is infinite.

DEFINITION 4.3.6 (WEYL GROUP). The Weyl group  $W$  associated to the Cartan matrix  $(a_{ij})$  is the quotient of the Braid group  $B$  by the normal subgroup generated by  $T_i T_i$  and  $T_i^{-1} T_i^{-1}$  ( $i \in [1, r]$ ), corresponding to relations  $T_i^2 = 1$  ( $i \in [1, r]$ ) in  $W$ :

$$W := B / \langle T_i^2, T_i^{-2} \mid i \in [1, r] \rangle.$$

I denote the image of  $T_i$  under the canonical map  $B \rightarrow W$  by  $s_i$ . For completeness I write down the relations that these generators of  $W$  satisfy:

$$\begin{aligned} s_i^2 &= 1 & (i \in [1, r]), \\ \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} &= \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}} & (i \neq j). \end{aligned}$$

The Weyl group  $W$  of the simple Lie algebra  $\mathfrak{g}$  is finite.  $W$  is an example of a Coxeter group.

PROPOSITION 4.3.7. *Let  $W$  be the Weyl group associated to  $(a_{ij})$ . The following action of  $W$  on  $P$*

$$s_i : x \mapsto x - \langle x, \alpha_i^\vee \rangle \alpha_i \quad (x \in P),$$

*is a group homomorphism  $W \rightarrow \text{Aut}(P) = GL(P)$ , i.e. this action defines a representation of  $W$  on  $P$ .*

PROOF. Since the map  $s_i$  is linear in  $x$ , it is clear that the action is automorphic. To prove that this defines a representation of  $W$ , it is necessary to check that the relations of  $W$  are satisfied on  $P$ . I show that  $s_i^2(x) = x$  ( $x \in P$ ).

$$\begin{aligned} s_i^2(x) &= s_i(x - \langle x, \alpha_i^\vee \rangle \alpha_i) \\ &= s_i(x) - \langle x, \alpha_i^\vee \rangle s_i(\alpha_i) \\ &= x, \end{aligned}$$

since  $s_i(\alpha_i) = -\alpha_i$ . Hence  $s_i$  acts as a reflection in  $P$ . The braid relations of  $W$  are similarly checked case by case.  $\square$

Note that the root sub-lattice  $Q$  of the weight lattice  $P$  is invariant under the action of  $W$ , i.e.  $W : Q \rightarrow Q \subset P$ . In particular  $W$  acts on the generators of  $Q$  as

$$s_j : \alpha_i \mapsto \alpha_i - \langle \alpha_i, \alpha_j^\vee \rangle \alpha_j.$$

Hence the restriction of the action of  $W$  to  $Q$ , gives a subrepresentation of  $W$  on  $Q$ .

LEMMA 4.3.8. *The following action of  $W$  on  $Q^\vee$  gives a representation of  $W$  on the coroot lattice.*

$$s_i : \alpha_j^\vee \mapsto \alpha_j^\vee - \langle \alpha_i, \alpha_j^\vee \rangle \alpha_i^\vee,$$

PROOF. The lemma is proved analogously to 4.3.7.  $\square$



**4.3.9. Roots.** Let  $\Pi := \{\alpha_i \mid i \in [1, r]\}$ , the simple roots of  $\mathfrak{g}$ . Define the root system of  $\mathfrak{g}$  corresponding to  $(a_{ij})$  to be  $R := W \cdot \Pi$  and  $R_+ := R \cap Q_+$  (the positive roots of  $\mathfrak{g}$ ).

LEMMA 4.3.10. *The pairing  $\langle \cdot, \cdot \rangle : P \times Q^\vee \rightarrow \mathbb{Z}$  is  $W$ -invariant:*

$$\langle s_i(x), y^\vee \rangle = \langle x, s_i(y^\vee) \rangle \quad (\forall x \in P, y^\vee \in Q^\vee, i \in [1, r]).$$

PROOF. Let  $x \in P$  and  $y^\vee \in Q^\vee$ .

$$\begin{aligned} \langle s_i(x), y^\vee \rangle &= \langle x - \langle x, \alpha_i^\vee \rangle \alpha_i, y^\vee \rangle \\ &= \langle x, y^\vee \rangle - \langle x, \alpha_i^\vee \rangle \langle \alpha_i, y^\vee \rangle \\ &= \langle x, y^\vee - \langle \alpha_i, y^\vee \rangle \alpha_i^\vee \rangle \\ &= \langle x, s_i(y^\vee) \rangle. \end{aligned}$$

□

COROLLARY 4.3.11.

$$\langle s_i(x), s_i(y^\vee) \rangle = \langle x, y^\vee \rangle \quad (\forall x \in P, y^\vee \in Q^\vee, i \in [1, r])$$

**4.3.12. Length function of  $W$ .** Let  $w \in W$ . The *length* of  $w$  is defined to be the smallest integer  $n \geq 0$  such that there exist  $i_1, i_2, \dots, i_n \in [1, r]$  with  $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ . I write  $l(w) = n$  for the length of  $w$ . A shortest expression  $s_{i_1} s_{i_2} \cdots s_{i_n}$  of  $w$  is called a *reduced expression* of  $w$ . Note that a reduced expression is not unique in general. The identity element  $1 \in W$  has length  $l(1) = 0$ . The generators of  $W$  have length  $l(s_i) = 1$ . There is a unique longest element of  $W$ , which is denoted  $w_0$ . It has the property that  $w_0 : R_+ \rightarrow -R_+$  and  $w_0^2 = 1$ .

#### 4.4. The Hopf algebra $\mathcal{U}_q(\mathfrak{g})$

NOTATION. Denote by  $\mathbb{C}[q, q^{-1}]$  the ring of polynomials in the indeterminate  $q$  and its inverse  $q^{-1}$ . Denote by  $\mathbb{C}(q)$  the quotient field of  $\mathbb{C}[q, q^{-1}]$ . Define  $q_i := q^{d_i}$ . In  $\mathbb{C}(q)$  I introduce the following standard notation

$$\begin{aligned} [m]_q &:= \frac{q^m - q^{-m}}{q - q^{-1}} & (m \in \mathbb{Z}), \\ [m]! &:= [m]_q [m-1]_q \cdots [1]_q & (m > 0), \\ [0]_q! &:= 1, \\ \begin{bmatrix} m \\ n \end{bmatrix}_q &:= \frac{[m]_q!}{[n]_q! [m-n]_q!} & (m \geq n \geq 0). \end{aligned}$$

Let  $m \in \mathbb{N}$ , such that  $m > 1$ . Note the following identity, which holds in  $\mathbb{Z}[q, q^{-1}] \subset \mathbb{C}[q, q^{-1}] \subset \mathbb{C}(q)$

$$q^m - q^{-m} = (q - q^{-1})(q^{m-1} + q^{m-3} + \cdots + q^{3-m} + q^{1-m}).$$

From it follows that  $[m]_q, [m]_q! \in \mathbb{Z}[q, q^{-1}]$ . It is also well known [Lus88] that  $\begin{bmatrix} m \\ n \end{bmatrix}_q \in \mathbb{Z}[q, q^{-1}]$ .

I come now to the definition of the quantum universal enveloping algebra of a Lie algebra  $\mathfrak{g}$ .

**DEFINITION 4.4.1.** Let  $\mathfrak{g}$  be a simple finite Lie algebra with Cartan matrix  $(a_{ij})$  ( $i, j \in [1, r]$ ). The *quantum universal enveloping algebra*  $\mathcal{U}_q(\mathfrak{g})$  of  $\mathfrak{g}$  is the associative unital  $\mathbb{C}(q)$ -algebra with generators  $\{k_i^{\pm 1}, e_i, f_i \mid i \in [1, r]\}$  satisfying the relations

$$\begin{aligned} k_i \cdot k_i^{-1} &= 1 = k_i^{-1} \cdot k_i, & k_i \cdot k_j &= k_j \cdot k_i, \\ k_i \cdot e_j \cdot k_i^{-1} &= q_i^{a_{ij}} e_j, & k_i \cdot f_j \cdot k_i^{-1} &= q_i^{-a_{ij}} f_j, \\ e_i \cdot f_j - f_j \cdot e_i &= \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} e_i^n \cdot e_j \cdot e_i^{1-a_{ij}-n} &= 0 \quad (i \neq j), \\ \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} f_i^n \cdot f_j \cdot f_i^{1-a_{ij}-n} &= 0 \quad (i \neq j). \end{aligned}$$

The last two sets of relations are called the quantum Serre relations.

**4.4.2.** The quantum enveloping algebra  $\mathcal{U}_q(\mathfrak{g})$  can be endowed with the structure of a Hopf algebra.

**PROPOSITION 4.4.3.** *The following maps, when extended homomorphically to the whole of  $\mathcal{U}_q(\mathfrak{g})$ , make  $(\mathcal{U}_q(\mathfrak{g}), \cdot, \mathbf{1}, \Delta, \epsilon, S; \mathbb{C}(q))$  a Hopf algebra.*

*The coproduct map  $\Delta : \mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$  is*

$$\begin{aligned} \Delta : k_i &\mapsto k_i \otimes k_i, \\ \Delta : e_i &\mapsto e_i \otimes \mathbf{1} + k_i \otimes e_i, \\ \Delta : f_i &\mapsto f_i \otimes k_i^{-1} + \mathbf{1} \otimes f_i. \end{aligned}$$

*The antipode map  $S : \mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g})$  is*

$$S : k_i \mapsto k_i^{-1}, \quad S : e_i \mapsto -k_i^{-1} e_i, \quad S : f_i \mapsto -f_i k_i.$$

*The counit map  $\epsilon : \mathcal{U}_q(\mathfrak{g}) \rightarrow \mathbb{C}(q)$  is*

$$\epsilon : k_i \mapsto 1, \quad \epsilon : e_i \mapsto 0, \quad \epsilon : f_i \mapsto 0.$$

**4.4.4.** There is a  $\mathbb{C}$ -algebra anti-automorphism  $\omega : \mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g})$

$$\begin{aligned}\omega : k_i &\mapsto k_i^{-1}, & \omega : q &\mapsto q^{-1}, \\ \omega : e_i &\mapsto f_i, & \omega : f_i &\mapsto e_i.\end{aligned}$$

Also there is a  $\mathbb{C}$ -algebra automorphism  $\varphi : \mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g})$

$$\begin{aligned}\varphi : k_i &\mapsto k_i, & \varphi : q &\mapsto q^{-1}, \\ \varphi : e_i &\mapsto f_i, & \varphi : f_i &\mapsto e_i.\end{aligned}$$

**4.4.5.** Let  $\mathcal{U}_q(\mathfrak{n}_+)$ ,  $\mathcal{U}_q(\mathfrak{h})$  and  $\mathcal{U}_q(\mathfrak{n}_-)$  be the subalgebras of  $\mathcal{U}_q(\mathfrak{g})$  generated by  $\{e_i \mid i \in [1, r]\}$ ,  $\{k_i, k_i^{-1} \mid i \in [1, r]\}$  and  $\{f_i \mid i \in [1, r]\}$  respectively. From the defining relations of  $\mathcal{U}_q(\mathfrak{g})$ , it follows that any element of  $\mathcal{U}_q(\mathfrak{g})$  can be written as a sum of monomials ordered such that elements of  $\mathcal{U}_q(\mathfrak{n}_-)$  appear on the left, elements of  $\mathcal{U}_q(\mathfrak{h})$  in the middle and elements of  $\mathcal{U}_q(\mathfrak{n}_+)$  on right in each monomial. Therefore  $\mathcal{U}_q(\mathfrak{g}) = \mathcal{U}_q(\mathfrak{n}_-)\mathcal{U}_q(\mathfrak{h})\mathcal{U}_q(\mathfrak{n}_+)$ ,

**4.4.6.** Denote by  $\mathcal{A}$  the ring  $\mathbb{C}[q, q^{-1}]$ . I define the elements  $h_i$  in  $\mathcal{U}_q(\mathfrak{g})$  to be

$$h_i := \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}.$$

Define the  $\mathcal{A}$ -subalgebra  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$  of  $\mathcal{U}_q(\mathfrak{g})$ , to be the subalgebra generated over  $\mathcal{A}$  by the elements in  $\{k_i^{\pm 1}, e_i, f_i, h_i \mid i \in [1, r]\}$ . The generators  $\{e_i, f_i, k_i^{\pm 1} \mid i \in [1, r]\}$  satisfy the relations in 4.4.1, except that the relation between  $e_i$  and  $f_j$  is replaced by

$$e_i \cdot f_j - f_j \cdot e_i = \delta_{ij} h_i,$$

and from the definition of  $h_i$  there is the relation

$$(q_i - q_i^{-1})h_i = k_i - k_i^{-1}.$$

Then  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$  is a Hopf subalgebra of  $\mathcal{U}_q(\mathfrak{g})$ . The coproduct, antipode and counit maps act on  $h_i$  as

$$\begin{aligned}\Delta(h_i) &= h_i \otimes k_i + k_i^{-1} \otimes h_i, \\ S(h_i) &= -h_i, \\ \epsilon(h_i) &= 0.\end{aligned}$$

NOTATION. For  $\epsilon \in \mathbb{C}^\times$ , define  $\epsilon_i := \epsilon^{d_i}$ . Denote by  $[n]_\epsilon$  and  $\begin{bmatrix} n \\ m \end{bmatrix}_\epsilon$  in  $\mathbb{C}$ , the numbers  $[n]_q$  and  $\begin{bmatrix} n \\ m \end{bmatrix}_q$  in  $\mathbb{C}(q)$  with  $q$  replaced by  $\epsilon$ .

**4.4.7.** Let  $\epsilon \in \mathbb{C}^\times$ . Then I define the *specialisation* of  $\mathcal{U}_A(\mathfrak{g})$  at  $q = \epsilon$  to be the  $\mathbb{C}$ -algebra  $\mathcal{U}_\epsilon(\mathfrak{g}) := \mathcal{U}_A(\mathfrak{g}) / \langle q - \epsilon \rangle$ .

**PROPOSITION 4.4.8.** (compare [DCK90, 1.5]) *Let  $\mathcal{U}_1(\mathfrak{g})$  be the specialisation at  $\epsilon = 1$ . The quotient algebra  $\mathcal{U}_1(\mathfrak{g}) / \langle k_i - 1, k_i^{-1} - 1 \mid i \in [1, r] \rangle$  is Hopf algebra isomorphic to the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  of the Lie algebra  $\mathfrak{g}$  defined in 4.2.2.*

**PROOF.** The  $\mathbb{C}$ -algebra isomorphism is given by

$$\begin{aligned} e_i &\mapsto \bar{e}_i, & f_i &\mapsto \bar{f}_i, \\ h_i &\mapsto \bar{h}_i. \end{aligned}$$

From inspection of the Hopf algebra maps of  $\mathcal{U}_1(\mathfrak{g})$ , it is clear that in the quotient algebra they reduce to the standard Hopf algebra maps of a universal enveloping algebra (see 2.7.8).  $\square$

#### 4.5. Braid group automorphisms of $\mathcal{U}_q(\mathfrak{g})$

**NOTATION.** I define for each  $n \in \mathbb{N}$  the following elements in  $\mathcal{U}_q(\mathfrak{g})$

$$e_i^{(n)} := \frac{e_i^n}{[n]_{q_i}!} \quad \text{and} \quad f_i^{(n)} := \frac{f_i^n}{[n]_{q_i}!}.$$

I quote the following well-known result, that appears for instance in [DCKP92, 2.2] and [Lus93, 2.1.2]

**THEOREM 4.5.1.** *Let  $B$  be the braid group and  $W$  the Weyl group of  $\mathfrak{g}$ . There is a unique map  $\phi : W \rightarrow B$ , such that  $\phi : 1 \mapsto 1$ ,  $\phi : s_i \mapsto T_i$  and  $\phi(w w') = \phi(w) \phi(w')$  when  $l(w w') = l(w) + l(w')$  ( $w, w' \in W$ ).*

**COROLLARY 4.5.2.** *Let  $s_{i_1} s_{i_2} \cdots s_{i_n}$  and  $s_{i'_1} s_{i'_2} \cdots s_{i'_n}$  be two reduced expressions of  $w \in W$ . Then  $T_{i_1} T_{i_2} \cdots T_{i_n} = T_{i'_1} T_{i'_2} \cdots T_{i'_n}$  in  $B$ . Write  $T_w$  for  $\phi(w)$  ( $w \in W$ ). The map  $\phi : w \mapsto T_w$  is well defined, since it is independent of the form of the reduced expression of  $w$ .*

I introduce Lusztig's braid group automorphisms  $T_i$  of  $\mathcal{U}_q(\mathfrak{g})$  [Lus88, Lus90b, Lus90c].

**THEOREM 4.5.3.** *Let  $B$  be the braid group associated to  $\mathfrak{g}$ . There is a representation of  $B$  on  $\mathcal{U}_q(\mathfrak{g})$  as a group of automorphisms:*

$$\begin{aligned} T_i &: k_i \mapsto k_i^{-1}, \\ T_i &: k_j \mapsto k_j k_i^{-a_{ij}} \quad (i \neq j), \\ T_i &: e_i \mapsto -k_i f_i, \\ T_i &: e_j \mapsto \sum_{n=0}^{-a_{ij}} (-1)^n q_i^{a_{ij}+n} e_i^{(n)} e_j e_i^{(-a_{ij}-n)} \quad (i \neq j), \\ T_i &: f_i \mapsto -k_i^{-1} e_i, \\ T_i &: f_j \mapsto \sum_{n=0}^{-a_{ij}} (-1)^n q_i^{-a_{ij}-n} f_i^{(-a_{ij}-n)} f_j f_i^{(n)} \quad (i \neq j). \end{aligned}$$

**PROOF.** The proof of the theorem is rather lengthy. It can be found in [Lus93].  $\square$

Note that the automorphisms commute with the anti-automorphism  $\omega$

$$T_i \circ \omega = \omega \circ T_i.$$

The action of the inverse elements is given by

$$T_i^{-1} = \varphi \circ T_i \circ \varphi^{-1}.$$

**REMARK 4.5.4.** [DCK90, 1.6] Note that  $T_i e_j = \text{ad}(-e_i^{(-a_{ij})}) e_j$  ( $i \neq j$ ) and  $T_i f_j = \omega(T_i e_j)$ .

**4.5.5.** Let  $w_0 \in W$  be the longest element of the Weyl group  $W$ . Fix a reduced expression  $s_{i_1} s_{i_2} \cdots s_{i_N}$  of  $w_0$ . Then the set

$$\alpha_{i_1}, \quad s_{i_1} \alpha_{i_2}, \quad s_{i_1} s_{i_2} \alpha_{i_3}, \quad \dots, \quad s_{i_1} s_{i_2} \cdots s_{i_{N-1}} \alpha_N$$

is in bijective correspondence with the set of positive roots  $R_+$  of  $\mathfrak{g}$  and fixes an ordering of  $R_+$ . Define  $\beta_k := s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \alpha_k$ . So  $R_+ = \{\beta_k \mid k \in [1, N]\}$ .

**PROPOSITION 4.5.6.** *Let  $\beta = w(\alpha_i) \in R_+$ , for some  $w \in W$  and  $i \in [1, r]$ . Then  $T_w e_i \in \mathcal{U}_q(\mathfrak{n}_+)$  and  $T_w f_i \in \mathcal{U}_q(\mathfrak{n}_-)$ .*

**PROPOSITION 4.5.7.** [DCKP92, 2.3] *Let  $w \in W$  such that  $w(\alpha_i) = \alpha_j$  ( $i \neq j$ ). Then  $T_w e_i = e_j$ .*

#### 4.6. A Basis of $\mathcal{U}_q(\mathfrak{g})$

The following important result is due to Lusztig.

**PROPOSITION 4.6.1.** [Lus90c, §4] *Let  $s_{i_1}s_{i_2}\cdots s_{i_N}$  be a reduced expression of  $w_0$ , the longest element of  $W$ . Let  $\{\beta_k \mid k \in [1, N]\}$  be the ordered set of positive roots defined above. Then the positive root vectors defined by*

$$e_{\beta_k} := T_{i_1}T_{i_2}\cdots T_{i_{k-1}}e_{i_k} \quad (k \in [1, N])$$

*generate an ordered basis of  $\mathcal{U}_q(\mathfrak{n}_+)$  as a vector space*

$$\mathcal{U}_q(\mathfrak{n}_+) = \sum_{(m_1, \dots, m_N) \in \mathbb{N}^N} \mathbb{C}(q) e_{\beta_1}^{m_1} e_{\beta_2}^{m_2} \cdots e_{\beta_N}^{m_N} =: \sum_{m \in \mathbb{N}^N} \mathbb{C}(q) E^m.$$

*The corresponding negative root vectors are defined by  $f_{\beta_k} := \omega(e_{\beta_k})$  and generate an ordered basis of  $\mathcal{U}_q(\mathfrak{n}_-)$*

$$\mathcal{U}_q(\mathfrak{n}_-) = \sum_{(m_1, \dots, m_N) \in \mathbb{N}^N} \mathbb{C}(q) f_{\beta_N}^{m_N} f_{\beta_{N-1}}^{m_{N-1}} \cdots f_{\beta_1}^{m_1} =: \sum_{m \in \mathbb{N}^N} \mathbb{C}(q) F^m.$$

The positive and negative root vectors form a linearly independent basis of  $\mathcal{U}_q(\mathfrak{n}_+)$  and  $\mathcal{U}_q(\mathfrak{n}_-)$  respectively. This is proved [Lus90b, proposition 1.10] essentially by observing that at the specialisation  $q = 1$ , the induced  $\mathcal{U}(\mathfrak{g})$ -basis is linearly independent.

**4.6.2.** The set  $\{k_i, k_i^{-1} \mid i \in [1, r]\}$  generates a basis of  $\mathcal{U}_q(\mathfrak{h})$

$$\mathcal{U}_q(\mathfrak{h}) = \sum_{(m_1, m_2, \dots, m_r) \in \mathbb{Z}^r} \mathbb{C}(q) k_1^{m_1} k_2^{m_2} \cdots k_r^{m_r} =: \sum_{m \in \mathbb{Z}^r} \mathbb{C}(q) K^m.$$

So I have now a basis of  $\mathcal{U}_q(\mathfrak{g}) = \mathcal{U}_q(\mathfrak{n}_-)\mathcal{U}_q(\mathfrak{h})\mathcal{U}_q(\mathfrak{n}_+)$

$$\mathcal{U}_q(\mathfrak{g}) = \sum_{m_{\pm} \in \mathbb{N}^N, m_0 \in \mathbb{Z}^r} \mathbb{C}(q) F^{m_-} K^{m_0} E^{m_+}.$$

Note that different reduced expressions of the longest element  $w_0$  of  $W$  give inequivalent bases of  $\mathcal{U}_q(\mathfrak{n}_+)$  and  $\mathcal{U}_q(\mathfrak{n}_-)$ .

**REMARK 4.6.3.** Let  $\mathfrak{g}$  be nonsimply-laced. Then the braid group action of  $B$  on  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$  is not strictly well-defined, since the elements  $e_i^{(n)}$  and  $f_i^{(n)}$  ( $i \in [1, r]$  and  $n > 1$ ) are not in  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$ . (Lusztig avoids this problem by considering a different  $\mathcal{A}$ -subalgebra of  $\mathcal{U}_q(\mathfrak{g})$  which includes these elements as generators.) Another way to overcome this, is to extend the ring  $\mathcal{A}$  to  $\bar{\mathcal{A}}$  by elements  $\frac{1}{q_i^n - q_i^{-n}}$  ( $n \in \{1, 2, 3\}$  (or  $n \in \mathbb{Z}_{>0}$ ),  $i \in [1, r]$ ). One should then ‘localise’  $\bar{\mathcal{A}}$  at  $q = \epsilon$  (meaning remove from  $\bar{\mathcal{A}}$  the elements that have poles at  $q = \epsilon$ ), so that one can consider the specialisation of  $\mathcal{U}_{\bar{\mathcal{A}}}(\mathfrak{g})$  at  $q = \epsilon$ . I thank Jonathan Beck for explaining to me the idea of the localisation of a ring in an indeterminate.

**4.6.4.  $\mathcal{U}_\epsilon(\mathfrak{g})$  basis.** The action of the braid group  $B$  on  $\mathcal{U}_\epsilon(\mathfrak{g})$  defined by the maps in 4.5.3 with  $q_i$  replaced by  $\epsilon_i$  is well-defined. For each reduced expression of  $w_0$  the action of  $B$  on  $\mathcal{U}_\epsilon(\mathfrak{g})$  gives an ordered basis of  $\mathcal{U}_\epsilon(\mathfrak{n}_+)$  (and  $\mathcal{U}_\epsilon(\mathfrak{n}_-)$ ) over  $\mathbb{C}$  as a vector space analogous to the basis of  $\mathcal{U}_q(\mathfrak{n}_+)$  ( $\mathcal{U}_q(\mathfrak{n}_-)$ ) of 4.6.1. The basis of  $\mathcal{U}_\epsilon(\mathfrak{h})$  is given by

$$\mathcal{U}_\epsilon(\mathfrak{h}) = \sum_{\substack{(m_1, m_2, \dots, m_r) \in \mathbb{Z}^r \\ (m'_1, m'_2, \dots, m'_r) \in \mathbb{N}^r}} \mathbb{C} k_1^{m_1} k_2^{m_2} \cdots k_r^{m_r} h_1^{m'_1} \cdots h_r^{m'_r} =: \sum_{m \in \mathbb{Z}^r, m' \in \mathbb{N}^r} \mathbb{C} K^m H^{m'}.$$

Finally there is the corresponding basis of  $\mathcal{U}_\epsilon(\mathfrak{g})$

$$\mathcal{U}_\epsilon(\mathfrak{g}) = \sum_{m_\pm, m' \in \mathbb{N}^N, m_0 \in \mathbb{Z}^r} \mathbb{C} F^{m_-} K^{m_0} H^{m'} E^{m_+}.$$

Observe that the above analysis is valid also at  $\epsilon = 1$  and in the quotient  $\mathcal{U}_1(\mathfrak{g})/\langle k_i - 1 \rangle$ . In the latter algebra the braid action reduces to the classical braid group action of  $B$  on  $\mathcal{U}(\mathfrak{g})$  and allows the construction of a basis (a Poincaré-Birkhoff-Witt basis), which in an obvious notation borrowed from above reads

$$\mathcal{U}(\mathfrak{g}) = \sum_{m_\pm, m_0 \in \mathbb{N}^N} \mathbb{C} \bar{F}^{m_-} \bar{H}^{m_0} \bar{E}^{m_+}.$$

Of course in this case the bases obtained from the different reduced expressions of  $w_0$  are all equivalent.

**4.6.5. Example.**  $\mathfrak{sl}_3$  is the easiest example of a simple Lie algebra with a nontrivial root system. I briefly consider the construction of a basis of  $\mathcal{U}_q(\mathfrak{sl}_3)$  using the techniques described above. The Cartan matrix  $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$  and  $r = 2$ . Let  $\{\alpha_1, \alpha_2\}$  be the simple roots of  $\mathfrak{sl}_3$ . The Weyl group  $W$  is generated by  $\{s_1, s_2\}$  and they satisfy the braid relations  $s_1 s_2 s_1 = s_2 s_1 s_2$  and  $s_i^2 = 1$  ( $i \in \{1, 2\}$ ).  $W$  contains the elements  $\{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$ . The longest element of  $W$  is  $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ . The positive roots are  $R_+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$  and  $N = 3$ .

Consider then the reduced expression  $s_1 s_2 s_1$  of  $w_0$ . Then  $\beta_1 = \alpha_1$ ,  $\beta_2 = \alpha_1 + \alpha_2$  and  $\beta_3 = \alpha_2$ . Straightforward calculations then show that the root vectors of  $\mathcal{U}_q(\mathfrak{sl}_3)$  corresponding to this reduced expression of  $w_0$  are

$$\begin{aligned} e_{\beta_1} &= e_1, \\ e_{\beta_2} &= T_1 e_2 = -e_1 e_2 + q^{-1} e_2 e_1, \\ e_{\beta_3} &= T_1 T_2 e_1 = e_2, \\ f_{\beta_1} &= \omega(e_{\beta_1}) = f_1, \\ f_{\beta_2} &= \omega(e_{\beta_2}) = -f_2 f_1 + q f_1 f_2, \\ f_{\beta_3} &= \omega(e_{\beta_3}) = f_2. \end{aligned}$$

If I choose instead the other reduced expression  $s_2 s_1 s_2$ . Then  $\beta'_1 = \alpha_2$ ,  $\beta'_2 = \alpha_2 + \alpha_1$

and  $\beta'_3 = \alpha_1$  and the root vectors obtained are

$$\begin{aligned} e_{\beta'_1} &= e_2, \\ e_{\beta'_2} &= T_2 e_1 = -e_2 e_1 + q^{-1} e_1 e_2, \\ e_{\beta'_3} &= T_1 T_2 e_1 = e_1, \\ f_{\beta'_1} &= \omega(e_{\beta'_1}) = f_2, \\ f_{\beta'_2} &= \omega(e_{\beta'_2}) = -f_1 f_2 + q f_2 f_1, \\ f_{\beta'_3} &= \omega(e_{\beta'_3}) = f_1. \end{aligned}$$

Note that  $e_{\beta_2}$  and  $e_{\beta'_2}$  are not proportional, and  $f_{\beta_2}$  and  $f_{\beta'_2}$  are not proportional to each other. So inequivalent bases are obtained from the two reduced different reduced expressions of  $w_0$ . However at the specialisation  $q = 1$ ,  $e_{\beta_2} = -e_{\beta'_2}$  and  $f_{\beta_2} = -f_{\beta'_2}$ . The basis of  $\mathcal{U}_q(\mathfrak{sl}_3)$  corresponding to the reduced expression  $s_1 s_2 s_1$  of  $w_0$  reads

$$\mathcal{U}_q(\mathfrak{sl}_3) = \sum_{m^\pm \in \mathbb{N}^3, m^0 \in \mathbb{Z}^2} f_{\beta_3}^{m_3^-} f_{\beta_2}^{m_2^-} f_{\beta_1}^{m_1^-} k_1^{m_1^0} k_2^{m_2^0} e_{\beta_1}^{m_1^+} e_{\beta_2}^{m_2^+} f_{\beta_3}^{m_3^+}.$$

#### 4.7. Representations of $\mathcal{U}_q(\mathfrak{g})$

**4.7.1.** Let  $P$  be the weight lattice,  $Q$  the root lattice and  $(\cdot, \cdot) : P \times Q \rightarrow \mathbb{Z}$  the bilinear form associated to the Cartan matrix  $(a_{ij})$  (as defined in 4.3). Let  $\text{Hom}(Q, \mathbb{Z}_2)$  be the group of homomorphisms of  $Q$  into the group  $\{1, -1\}$ .

**LEMMA 4.7.2.** *Let  $\sigma \in \text{Hom}(Q, \mathbb{Z}_2)$ . Then the following map is an automorphism of  $\mathcal{U}_q(\mathfrak{g})$*

$$\begin{aligned} e_i &\mapsto \sigma(\alpha_i) e_i, \\ k_i &\mapsto \sigma(\alpha_i) k_i, \\ f_i &\mapsto f_i. \end{aligned}$$

**DEFINITION 4.7.3.** Fix  $\sigma \in \text{Hom}(Q, \mathbb{Z}_2)$  and  $\lambda \in P$ . Then the *Verma module*  $M^\sigma(\lambda)$  over  $\mathcal{U}_q(\mathfrak{g})$  with highest weight  $\lambda$  and twisting  $\sigma$  is defined to be the (unique)  $\mathbb{C}(q)$ -vector space generated by a vector  $v_\lambda$  such that

$$\begin{aligned} \mathcal{U}_q(\mathfrak{n}_+) \cdot v_\lambda &= 0, \\ k_i \cdot v_\lambda &= \sigma(\alpha_i) q^{(\lambda, \alpha_i)} v_\lambda, \\ M^\sigma(\lambda) &:= \mathcal{U}_q(\mathfrak{n}_-) \cdot v_\lambda = \sum_{m \in \mathbb{N}^N} F^m \cdot v_\lambda. \end{aligned}$$

The vector  $v_\lambda$  is called a highest weight vector of the module.



**4.7.4. Highest weight modules.** I can construct the Verma module  $M^\sigma(\lambda)$  in the following way. Consider the left ideal of  $\mathcal{U}_q(\mathfrak{g})$

$$J^\sigma(\lambda) := \sum_{i \in [1, r]} \mathcal{U}_q(\mathfrak{g}) e_i + \sum_{i \in [1, r]} \mathcal{U}_q(\mathfrak{g}) (k_i - \sigma(\alpha_i) q^{(\lambda, \alpha_i)}).$$

Then the quotient  $\mathcal{U}_q(\mathfrak{g})/J^\sigma(\lambda)$  is isomorphic to  $M^\sigma(\lambda)$  as a  $\mathcal{U}_q(\mathfrak{g})$ -module.

The  $\mathcal{U}_q(\mathfrak{g})$ -module  $M^\sigma(\lambda)$  is a highest weight module. Every quotient module of  $M^\sigma(\lambda)$  is a highest weight  $\mathcal{U}_q(\mathfrak{g})$ -module with highest weight  $\lambda$ .

In particular there exists a unique maximum submodule  $M'$  of  $M^\sigma(\lambda)$ . Therefore the quotient module  $L^\sigma(\lambda) := M^\sigma(\lambda)/M'$  is unique and irreducible.

Let  $V$  be a highest weight  $\mathcal{U}_q(\mathfrak{g})$ -module with highest weight  $\lambda$ . Define the subspace  $V_\mu$  of  $V$  ( $\mu \in P_+$ ) as

$$V_\mu := \{v \in V \mid k_i \cdot v = \sigma(\alpha_i) q^{(\lambda - \mu, \alpha_i)} v\}.$$

Then  $V$  admits the following weight space decomposition ( $P_+$ -gradation)

$$V = \bigoplus_{\mu \in P_+} V_\mu.$$

**DEFINITION 4.7.5.** Let  $V$  be a  $\mathcal{U}_q(\mathfrak{g})$ -module, with weight space decomposition.  $V$  is called *integrable*, if there exists a positive integer  $m_0 \in \mathbb{Z}_{>0}$ , such that  $e_i^{m_0} \cdot v = 0$  and  $f_i^{m_0} \cdot v = 0$  for all  $m \geq m_0$ ,  $i \in [1, r]$  and  $v \in V$ . (In this case the action of the Chevalley generators of  $\mathcal{U}_q(\mathfrak{g})$  on  $V$  is said to be locally nilpotent.)

Integrable modules [Lus88] of  $\mathcal{U}_q(\mathfrak{g})$  can be constructed as follows.

**LEMMA 4.7.6.** Let  $(m_i^+), (m_i^-) \in \mathbb{N}^r$  and  $\lambda \in P$ . Define a left ideal  $I_\lambda$  of  $\mathcal{U}_q(\mathfrak{g})$  by

$$I_\lambda := \sum_{i \in [1, r]} \mathcal{U}_q(\mathfrak{g}) e_i^{m_i^+ + 1} + \sum_{i \in [1, r]} \mathcal{U}_q(\mathfrak{g}) f_i^{m_i^- + 1} + \sum_{i \in [1, r]} \mathcal{U}_q(\mathfrak{g}) (k_i - q^{(\lambda, \alpha_i)}).$$

Then the quotient  $\mathcal{U}_q(\mathfrak{g})/I_\lambda$  is an integrable  $\mathcal{U}_q(\mathfrak{g})$ -module.

**PROOF.** See [Lus93, 3.5.3]. I sketch the proof. The idea is to show using the defining relations of  $\mathcal{U}_q(\mathfrak{g})$  that for every  $x \in \mathcal{U}_q(\mathfrak{g})$ ,  $f_i^N x = y f_i^{N'}$  ( $y \in \mathcal{U}_q(\mathfrak{g})$ ) where  $N' \geq N - c_x \in \mathbb{N}$  ( $c_x \in \mathbb{N}$  depending only  $x$ ). Then for  $N \geq m_i^- + 1 + c_x$  and  $x \in \mathcal{U}_q(\mathfrak{g})/I_\lambda$ ,  $f_i^N x = 0$ . There is a corresponding result with  $f_i$  replaced by  $e_i$ . This proves the lemma.  $\square$

**4.7.7. Trivial module.** As usual the counit map  $\epsilon : \mathcal{U}_q(\mathfrak{g}) \rightarrow \mathbb{C}(q)$  gives a trivial (one dimensional) representation of  $\mathcal{U}_q(\mathfrak{g})$  on  $\mathbb{C}(q)$ . (The representation is integrable.)

**4.7.8. Tensor product module.** Let  $V_1$  and  $V_2$  be two  $\mathcal{U}_q(\mathfrak{g})$ -modules. Then the coproduct map  $\Delta : \mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$  induces a  $\mathcal{U}_q(\mathfrak{g})$ -module structure on  $V_1 \otimes V_2$ . It can be shown [Lus93, 3.5.2] that the tensor product of two integrable  $\mathcal{U}_q(\mathfrak{g})$ -modules is integrable.

**4.7.9. Dual module.** Let  $V$  be a (left)  $\mathcal{U}_q(\mathfrak{g})$ -module and let  $V^*$  be the  $\mathbb{C}(q)$ -vector space dual to  $V$ , under a pairing  $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{C}(q)$ . Then the antipode map  $S$  induces a dual (left)  $\mathcal{U}_q(\mathfrak{g})$ -module

$$\langle x \cdot v^*, v \rangle := \langle v^*, S(x) \cdot v \rangle \quad (x \in \mathcal{U}_q(\mathfrak{g}), v \in V, v^* \in V^*).$$

The dual module to an integrable module is integrable.

**4.7.10. Right module.** Let  $\pi : \mathcal{U}_q(\mathfrak{g}) \rightarrow \text{End}(V)$  be a left representation, so that  $\pi(xy)v = \pi(x)\pi(y)v$  ( $x, y \in \mathcal{U}_q(\mathfrak{g}), v \in V$ ). Then the composition of the anti-automorphism  $\omega$  of  $\mathcal{U}_q(\mathfrak{g})$  with  $\pi$  gives a right representation  $\pi^\omega := \pi \circ \omega$  of  $\mathcal{U}_q(\mathfrak{g})$ , so that  $\pi^\omega(xy)v = \pi^\omega(y)\pi^\omega(x)v$  ( $x, y \in \mathcal{U}_q(\mathfrak{g}), v \in V$ ). Conversely if  $\pi'$  is a right representation then  $\pi' \circ \omega$  is a left representation. If  $\pi$  is an integrable representation, then  $\pi \circ \omega$  is an integrable representation.

**4.7.11.** The Verma module  $M_{\mathcal{A}}^\sigma(\lambda)$  over  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$  and the Verma module  $M_\epsilon^\sigma(\lambda)$  over  $\mathcal{U}_\epsilon(\mathfrak{g})$  are defined like  $M^\sigma(\lambda)$  in an obvious way.

#### 4.8. $\mathcal{U}_\epsilon(\mathfrak{g})$ at Roots of unity

In this section I consider the specialisation of  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$  at  $q = \epsilon$ , a primitive root of unity. First I present some identities, which will be required shortly.

LEMMA 4.8.1. [Jim85, §3] *Let  $m \in \mathbb{Z}_{>0}$  and  $i, j \in [1, r]$ . The following identities hold in  $\mathcal{U}_q(\mathfrak{g})$*

$$\begin{aligned} [e_i^m, f_j] &= \delta_{ij} [m]_{q_i} \frac{k_i q_i^{1-m} - k_i^{-1} q_i^{m-1}}{q_i - q_i^{-1}} e_i^{m-1}, \\ [e_i, f_j^m] &= \delta_{ij} [m]_{q_j} f_j^{m-1} \frac{k_j q_j^{1-m} - k_j^{-1} q_j^{m-1}}{q_j - q_j^{-1}}. \end{aligned}$$

PROOF. The first identity is proved by induction on  $m$ . The second then follows by applying the anti-automorphism  $\omega$ .  $\square$

LEMMA 4.8.2. [DCK90, 1.10] *Define the following elements in  $\mathcal{U}_q(\mathfrak{n}_+)$ :*

$$\begin{aligned} e_{ij} &:= T_j e_i && \text{if } a_{ij} < 0, \\ e_{iij} &:= \begin{cases} T_j T_i e_j & \text{if } a_{ij} = -2 \\ T_j T_i T_j e_i & \text{if } a_{ij} = -3, \end{cases} \\ e_{iiij} &:= T_j T_i T_j T_i e_j && \text{if } a_{ij} = -3. \end{aligned}$$

Let  $m \geq -a_{ij}$ . Then the following identity holds in  $\mathcal{U}_q(\mathfrak{n}_+)$

$$e_i^m e_j = \sum_{p=0}^{-a_{ij}} q_i^{(-a_{ij}-p)m+p} \frac{[m]_{q_i}!}{[m-p]_{q_i}!} \underbrace{e_{i\dots i j}}_{p \text{ times}} e_i^{m-p}.$$

A similar identity in the generators  $f_i$  is obtained in  $\mathcal{U}_q(\mathfrak{n}_-)$  by applying the anti-automorphism  $\omega$  to the above identity.

PROOF. The lemma follows from direct calculations that can be found in [Lus90c, §5].  $\square$

**4.8.3.** Fix a positive *odd* integer  $\ell \in \mathbb{Z}_{>0}$ , such that  $\ell > d_i$  ( $\forall i \in [1, r]$ ). Let  $\epsilon$  be a primitive (odd)  $\ell$ -th root of unity.

PROPOSITION 4.8.4. [DCK90, 3.1] *Let  $\{e_\beta, f_\beta \mid \beta \in R_+\}$  be the basis of the positive and negative root vectors of  $\mathcal{U}_\epsilon(\mathfrak{g})$ , introduced in 4.6.1. In  $\mathcal{U}_\epsilon(\mathfrak{g})$  at the odd root of unity  $\epsilon$  the following relations hold for all  $\alpha, \beta \in R_+$ ,  $i \in [1, r]$ :*

$$\begin{aligned} (1) \quad & k_i^\ell e_\beta = e_\beta k_i^\ell, & k_i^\ell f_\beta &= f_\beta k_i^\ell, \\ (2) \quad & e_\alpha^\ell f_\beta = f_\beta e_\alpha^\ell, & f_\alpha^\ell e_\beta &= e_\beta f_\alpha^\ell, \\ (3) \quad & e_\alpha^\ell e_\beta = e_\beta e_\alpha^\ell, & f_\alpha^\ell f_\beta &= f_\beta f_\alpha^\ell. \end{aligned}$$

PROOF. The relations in (1) for  $k_i^\ell$  follow immediately from the corresponding defining relations of  $\mathcal{U}_q(\mathfrak{g})$ . Consider now (2) and (3). First let  $\alpha = \alpha_i$  and  $\beta = \alpha_j$ . In this case the relations in (2) follow from lemma 4.8.1 and those in (3) from lemma 4.8.2. But then since  $\{e_{\alpha_j} \mid j \in [1, r]\}$  generates  $\mathcal{U}_q(\mathfrak{n}_+)$  and  $\{f_{\alpha_j} \mid j \in [1, r]\}$  generates  $\mathcal{U}_q(\mathfrak{n}_-)$ , it follows that (2) and (3) are true for all  $\alpha_i \in \Pi$  and all  $\beta \in R_+$ . Next I apply a braid group automorphism  $T_w$  ( $w \in W$ ) to the relations in (2) and (3) when  $\alpha = \alpha_i$  and  $\beta \in R_+$  and this proves that the relations are in fact true for all  $\alpha = w(\alpha_i)$  ( $\forall w \in W$ ) and  $\beta \in R_+$ . Therefore the relations hold for all  $\alpha, \beta \in R_+$  and the proposition is proved.  $\square$

NOTATION. Let  $\mathcal{Z}_\epsilon$  denote the centre of  $\mathcal{U}_\epsilon(\mathfrak{g})$ .

COROLLARY 4.8.5. *At the root of unity  $\epsilon$ , the following elements lie in the centre  $\mathcal{Z}_\epsilon$  of  $\mathcal{U}_\epsilon(\mathfrak{g})$ :*

$$e_\alpha^\ell, k_i^\ell, f_\alpha^\ell \quad (\forall \alpha \in R_+, i \in [1, r]).$$

**4.8.6. Diagonal modules.** Consider now a Verma module  $M_\epsilon^\sigma(\lambda)$  over  $\mathcal{U}_q(\mathfrak{g})$  at the root of unity  $\epsilon$ , generated by  $v_\lambda$ . Let  $\alpha \in R_+$ . Note that the vector  $f_\alpha^\ell \cdot v_\lambda$  is singular (primitive) in  $M_\epsilon^\sigma(\lambda)$ :

$$\mathcal{U}_\epsilon(\mathfrak{n}_+) \cdot f_\alpha^\ell \cdot v_\lambda = 0 \quad (\forall \alpha \in R_+),$$

since  $f_\alpha^\ell \in \mathcal{Z}_\epsilon$ . Therefore each vector  $f_\alpha^\ell \cdot v_\lambda$  ( $\alpha \in R_+$ ) generates a  $\mathcal{U}_\epsilon(\mathfrak{g})$ -submodule of  $M_\epsilon^\sigma(\lambda)$ . Define

$$\bar{M}_\epsilon^\sigma(\lambda) := M_\epsilon^\sigma(\lambda) / \left( \sum_{\alpha \in R_+} \mathcal{U}_q(\mathfrak{g}) \cdot f_\alpha^\ell \cdot v_\lambda \right).$$

In [DCK90, 3.2] the  $\mathcal{U}_\epsilon(\mathfrak{g})$ -module  $\bar{M}_\epsilon^\sigma(\lambda)$  is called diagonal.

PROPOSITION 4.8.7. *Let  $\mathbb{Z}_\ell := \mathbb{Z}/\ell\mathbb{Z}$ . The  $\mathcal{U}_\epsilon(\mathfrak{g})$ -module  $\bar{M}_\epsilon^\sigma(\lambda)$  is finite dimensional. It has the following basis over  $\mathbb{C}$*

$$\bar{M}_\epsilon^\sigma(\lambda) = \sum_{m \in \mathbb{Z}_\ell^N} \mathbb{C} F^m v_\lambda.$$

REMARK 4.8.8. Like the Verma module  $M_\epsilon^\sigma(\lambda)$ , the diagonal module  $\bar{M}_\epsilon^\sigma(\lambda)$  can of course be constructed as a quotient by a left ideal of  $\mathcal{U}_\epsilon(\mathfrak{g})$  in a similar way to 4.7.4.

**4.8.9. The centre.** Let  $x_\alpha := e_\alpha^\ell$ ,  $y_\alpha := f_\alpha^\ell$  ( $\alpha \in R_+$ ) and  $z_i^{\pm 1} := k_i^{\pm \ell}$  ( $i \in [1, r]$ ). Define  $\mathcal{Z}_0^+$  ( $\mathcal{Z}_0^-$ ) to be the central subalgebra of  $\mathcal{U}_\epsilon(\mathfrak{n}_+)$  ( $\mathcal{U}_\epsilon(\mathfrak{n}_-)$ ) generated by  $\{x_\alpha \mid \alpha \in R_+\}$  (respectively  $\{y_\alpha \mid \alpha \in R_+\}$ ). Define  $\mathcal{Z}_0^0$  to be the subalgebra of  $\mathcal{U}_\epsilon(\mathfrak{h})$  generated by  $\{z_i^{\pm 1} \mid i \in [1, r]\}$ . Let  $\mathcal{Z}_0$  be the subalgebra of  $\mathcal{Z}_\epsilon$  generated by the subalgebras  $\mathcal{Z}_0^+$ ,  $\mathcal{Z}_0^-$  and  $\mathcal{Z}_0^0$ . Note that  $\mathcal{Z}_0$  is a proper subalgebra of  $\mathcal{Z}_\epsilon$ , since  $\mathcal{U}_q(\mathfrak{g})$  has one or more  $q$ -analogue Casimir elements, that generate  $\mathcal{Z}_\epsilon$  when  $\epsilon$  is not a root of unity.

LEMMA 4.8.10. *The algebra  $\mathcal{U}_\epsilon(\mathfrak{g})$  is finite dimensional over  $\mathcal{Z}_0$  and*

$$\{F^{m_-} K^{m_0} E^{m_+} \mid m_\pm \in \mathbb{Z}_\ell^N, m_0 \in \mathbb{Z}_\ell^r\}$$

*forms a basis of  $\mathcal{U}_\epsilon(\mathfrak{g})$  over  $\mathcal{Z}_0$  as a free module. So  $\dim_{\mathcal{Z}_0} \mathcal{U}_\epsilon(\mathfrak{g}) = \ell^{\dim \mathfrak{g}}$ .*

NOTATION. Let  $\alpha \in R_+$ , such that  $\alpha = \sum_{i \in [1, r]} a_i \alpha_i$ . Define  $k_\alpha := \prod_{i \in [1, r]} k_i^{a_i}$ .

**4.8.11. Triangular modules.** Let  $\lambda \in P$  and let  $\nu$  be an algebra homomorphism  $\nu : \mathcal{Z}_0^- \rightarrow \mathbb{C}$ . Then define the *triangular module*  $\bar{M}_\epsilon^\sigma(\lambda, \nu)$  over  $\mathcal{U}_\epsilon(\mathfrak{g})$  to be the quotient of  $\mathcal{U}_\epsilon(\mathfrak{g})$  by the left ideal generated by  $\{e_i, k_i^{\pm 1} - \sigma(\alpha_i) q^{\pm(\lambda, \alpha_i)}, y_\alpha - \nu(y_\alpha) \mid i \in [1, r], \alpha \in R_+\}$ . Denote by  $v_\lambda$  the image of 1 in  $\bar{M}_\epsilon^\sigma(\lambda, \nu)$ . The elements  $F^m \cdot v_\lambda$  ( $m \in \mathbb{Z}_\ell^n$ ) form a (finite dimensional) basis of  $\bar{M}_\epsilon^\sigma(\lambda, \nu)$ .

Let  $\alpha \in R_+$ . When  $\nu(y_\alpha) = 0$ , I say that  $f_\alpha$  acts nilpotently in  $\bar{M}_\epsilon^\sigma(\lambda, \nu)$ . In this case the  $\mathcal{U}_\epsilon(\mathfrak{sl}_2)$ -submodule generated by  $\{e_\alpha, f_\alpha, k_\alpha^{\pm 1}\}$  is called nilpotent. When  $\nu(y_\alpha) \neq 0$ , I say that  $f_\alpha$  acts cyclically (periodically). In this case the  $\mathcal{U}_\epsilon(\mathfrak{sl}_2)$ -submodule corresponding to the root  $\alpha$  is called semicyclic (semiperiodic), since  $f_\alpha$  acts cyclically but  $e_\alpha$  acts nilpotently.

A  $\mathcal{U}_\epsilon(\mathfrak{sl}_2)$ -module is called cyclic (periodic) when both its Chevalley generators  $e$  and  $f$  act in it cyclically. By definition there are no cyclic  $\mathcal{U}_\epsilon(\mathfrak{sl}_2)$ -submodules in a triangular module.

Note that the triangular  $\mathcal{U}_\epsilon(\mathfrak{g})$ -module  $\bar{M}_\epsilon^\sigma(\lambda, \nu)$  contains the diagonal module  $\bar{M}_\epsilon^\sigma(\lambda)$  as a special case: in the case  $\nu = 0$ , then  $\bar{M}_\epsilon^\sigma(\lambda, 0) = \bar{M}_\epsilon^\sigma(\lambda)$ .

**4.8.12. Central characters.** Let  $(V, \pi)$  be an irreducible (finite dimensional) representation of  $\mathcal{U}_\epsilon(\mathfrak{g})$  (at the root of unity  $\epsilon$ ). Then by Schur's lemma

$$\pi(x) \cdot v = \chi^\pi(x) v \quad (x \in \mathcal{Z}_\epsilon, v \in V, \chi^\pi(x) \in \mathbb{C}).$$

The map  $\chi^\pi : \mathcal{Z}_\epsilon \rightarrow \mathbb{C}$  is called the central character of the representation  $\pi$ . Note in particular that  $\chi^\pi(z_i^{\pm 1}) \neq 0$  ( $i \in [1, r]$ ).

If a  $\mathcal{U}_\epsilon(\mathfrak{g})$ -module is diagonal, then  $\chi : \mathcal{Z}_0^\pm \rightarrow 0$ . For a triangular module the central character maps  $\chi : \mathcal{Z}_0^+ \rightarrow 0$ , but  $\chi(y_\alpha) \neq 0$  ( $\alpha \in R_+$ ). In a completely cyclic (periodic) module  $\chi(x_\alpha) \neq 0$  and  $\chi(y_\alpha) \neq 0$  ( $\alpha \in R_+$ ).

#### 4.9. $\mathcal{U}_q(\mathfrak{sl}_2)$

In this section I consider as an example the case of  $\mathfrak{sl}_2$ .

**4.9.1.** The Lie algebra  $\mathfrak{sl}_2$  has a  $1 \times 1$  Cartan matrix (2) and only a single positive root  $\alpha$ . The weight lattice is  $P = \mathbb{Z}$  ( $P_+ = \mathbb{N}$ ) and the root lattice  $Q = 2\mathbb{Z}$  ( $Q_+ = 2\mathbb{N}$ ). The Weyl group of  $\mathfrak{sl}_2$  has only two elements: the identity and the reflection  $\alpha \leftrightarrow -\alpha$ . Therefore it is very simple to do calculations in  $\mathfrak{sl}_2$ , since it is the smallest simple Lie algebra and it has a minimal root system ( $N = 1$ ).

The quantum group  $\mathcal{U}_q(\mathfrak{sl}_2)$  is generated over  $\mathbb{C}(q)$  by  $\{e, f, k^{\pm 1}\}$ , satisfying the relations

$$\begin{aligned} k \cdot e \cdot k^{-1} &= q^2 e, & k \cdot f \cdot k^{-1} &= q^{-2} f, \\ e \cdot f - f \cdot e &= \frac{k - k^{-1}}{q - q^{-1}}, & k \cdot k^{-1} &= 1 = k^{-1} \cdot k. \end{aligned}$$

The algebra has the following basis

$$\mathcal{U}_q(\mathfrak{sl}_2) := \sum_{m_{\pm} \in \mathbb{N}, m_0 \in \mathbb{Z}} \mathbb{C}(q) f^{m_-} k^{m_0} e^{m_+}.$$

**4.9.2.** The simplest nontrivial representation of  $\mathcal{U}_q(\mathfrak{sl}_2)$  is the 2-dimensional vector representation  $(\mathbb{C}(q)^2, \pi)$ :

$$\pi(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi(k) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad \pi(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

**4.9.3. Spin- $\frac{j}{2}$  representations.** Let  $\sigma \in \{1, -1\}$  and  $j \in P_+ = \mathbb{N}$ . Let the  $\mathbb{C}(q)$ -vector space  $V^\sigma(j) := \sum_{m=0}^j \mathbb{C} v_m$ , with basis  $\{v_0, \dots, v_j\}$ . Define a representation of  $\mathcal{U}_q(\mathfrak{g})$  on  $V_j^\sigma$  by:

$$\begin{aligned} k \cdot v_m &= \sigma q^{j-2m} v_m, \\ e \cdot v_m &= \sigma [j - m + 1]_q v_{m-1}, \\ f \cdot v_m &= [m + 1]_q v_{m+1}, \end{aligned}$$

and  $v_m := 0$  when  $m \notin [0, j]$ .

LEMMA 4.9.4. *The  $\mathcal{U}_q(\mathfrak{sl}_2)$ -module  $V^\sigma(j)$  is irreducible.*

PROOF. The lemma is true since the coefficients of the  $e$  and  $f$  actions ( $[m + 1]_q$  and  $[j - m + 1]_q$ ) are nonzero for  $m \in [0, j]$ : so there are no singular vectors in  $V^\sigma(j)$ .  $\square$

For any irreducible  $n$ -dimensional representation  $V$  of  $\mathcal{U}_q(\mathfrak{sl}_2)$ ,  $\sigma$  can be chosen in  $\{1, -1\}$  such that  $V$  is equivalent to  $V^\sigma(n - 1)$  as a  $\mathcal{U}_q(\mathfrak{sl}_2)$ -module.

**4.9.5. Verma module.** Consider the Verma module  $M^\sigma(\lambda)$  of  $\mathcal{U}_q(\mathfrak{sl}_2)$  generated by the highest weight vector  $v_\lambda$ , with highest weight  $\lambda \in P$  ( $k \cdot v_\lambda = \sigma q^\lambda v_\lambda$  and  $e \cdot v_\lambda = 0$ ). Then the vectors  $v_m := \frac{1}{[m]_q!} f^m \cdot v_\lambda$  ( $m \in \mathbb{Z}_{>0}$ ) and  $v_0 := v_\lambda$  are a basis of  $M^\sigma(\lambda)$ . The generators act on  $M^\sigma(\lambda)$  as

$$\begin{aligned} f \cdot v_m &= [m + 1]_q v_{m+1}, \\ k \cdot v_m &= \sigma q^{\lambda-2m} v_m, \\ e \cdot v_m &= \sigma [\lambda - m + 1] v_{m-1}. \end{aligned}$$

Observe that if  $\lambda \geq 0$ , then  $v' := v_{j'}$  with  $j' = \lambda + 1$  is a primitive (singular) vector in  $M^\sigma(\lambda)$  ( $e \cdot v' = 0$ ) and it is the first primitive vector in  $M^\sigma(\lambda)$  below  $v_\lambda$ . Therefore  $L^\sigma(\lambda) := M^\sigma(\lambda)/(\mathcal{U}_q(\mathfrak{sl}_2) \cdot v')$  is a finite  $j'$  dimensional irreducible representation of  $\mathcal{U}_q(\mathfrak{sl}_2)$ . In fact  $L^\sigma(\lambda)$  is equivalent to the  $\mathcal{U}_q(\mathfrak{sl}_2)$ -module  $V_\lambda^\sigma$  described above.  $L(\lambda) \simeq \mathcal{U}_q(\mathfrak{sl}_2)/\langle e, f^{\lambda+1}, k - \sigma q^\lambda \rangle_L$ . (The notation  $\langle \cdot \rangle_L$  denotes a left ideal in  $\mathcal{U}_q(\mathfrak{sl}_2)$ .) Clearly these irreducible finite dimensional representations are integrable  $\mathcal{U}_q(\mathfrak{sl}_2)$ -modules.

**4.9.6. Root of unity.** Let  $\epsilon \in \mathbb{C}^\times$ . Define as before the  $\mathcal{A}$ -subalgebra  $\mathcal{U}_\mathcal{A}(\mathfrak{sl}_2)$  and its specialisation  $\mathcal{U}_\epsilon(\mathfrak{sl}_2)$  at  $q = \epsilon$ .

Fix a positive integer  $\ell \in \mathbb{Z}_{>0}$ , such that  $\ell > 2$ . Let  $\epsilon$  be a primitive  $\ell$ -th root of unity. Define

$$\ell' = \begin{cases} \ell & \text{if } \ell \text{ is odd,} \\ \frac{\ell}{2} & \text{if } \ell \text{ is even.} \end{cases}$$

LEMMA 4.9.7. *At the root of unity  $\epsilon$ , the following elements are in the centre  $\mathcal{Z}_\epsilon$  of  $\mathcal{U}_\epsilon(\mathfrak{sl}_2)$*

$$e^{\ell'}, k^{\pm \ell'}, f^{\ell'}.$$

**4.9.8.** Let  $x := e^{\ell'}$ ,  $z^{\pm 1} := k^{\pm \ell'}$  and  $y := f^{\ell'}$ . They generate a subalgebra  $\mathcal{Z}_0$  of the centre  $\mathcal{Z}_\epsilon$ .

**4.9.9. Spin- $\frac{j}{2}$  representations at a root of unity.** Consider now the  $\mathcal{U}_\epsilon(\mathfrak{sl}_2)$ -module  $V^\sigma(j)$  at the root of unity  $\epsilon$ . If  $j < \ell'$ , then the module is irreducible. If  $j \geq \ell'$ , then the module contains one or more singular vectors  $\{v_p \mid p \in \{j+1 \bmod \ell'\} \cap [0, j]\}$  and is therefore reducible. Let  $\chi$  be the central character of the representation. Observe that  $\chi(x) = 0$  and  $\chi(y) = 0$ .

**4.9.10. Cyclic representations.** Every irreducible  $\ell'$ -dimensional  $\mathcal{U}_\epsilon(\mathfrak{sl}_2)$ -module, at the root of unity  $\epsilon$ , is isomorphic [DCK90, 4.2] for some  $(\zeta, a, b) \in \mathbb{C}^\times \times \mathbb{C}^2$  to the following representation  $V(\zeta, a, b)$  with basis  $\{v_m \mid m \in [0, \ell' - 1]\}$

$$\begin{aligned} k \cdot v_m &= \zeta \epsilon^{-2m} v_m & (m \in [0, \ell' - 1]), \\ f \cdot v_m &= v_{m+1} & (m \in [0, \ell' - 2]), \\ f \cdot v_{\ell'-1} &= b v_0, \\ e \cdot v_m &= \left( \frac{\zeta \epsilon^{1-m} - \zeta^{-1} \epsilon^{m-1}}{\epsilon - \epsilon^{-1}} [j]_\epsilon + ab \right) v_{m-1} & (m \in [1, \ell' - 1]), \\ e \cdot v_0 &= a v_{\ell'-1}. \end{aligned}$$

Note that the  $\mathcal{Z}_0$  character  $\chi$  is

$$\begin{aligned} \chi(z) &= \zeta^{\ell'}, \\ \chi(x) &= a \prod_{j=1}^{\ell'-1} \left( \frac{\zeta \epsilon^{1-m} - \zeta^{-1} \epsilon^{m-1}}{\epsilon - \epsilon^{-1}} [j]_\epsilon + ab \right), \\ \chi(y) &= b. \end{aligned}$$

In particular  $\chi(x) = 0$ , if  $a = 0$ . Note that  $V(\sigma\epsilon^{\ell'-1}, 0, 0)$  is equivalent to  $V^\sigma(\ell' - 1)$ . In this case  $(a, b) = (0, 0)$  and the representation is called nilpotent.

In the case  $(a, b) \in \mathbb{C}^\times \times \{0\} \cup \{0\} \times \mathbb{C}^\times$ , the representation  $V(\zeta, a, b)$  is called semicyclic (semiperiodic).

In the case when  $(a, b) \in \mathbb{C}^\times \times \mathbb{C}^\times$ , the representation  $V(\zeta, a, b)$  is called cyclic (periodic).

The representation  $V(\zeta, a, b)$  is fundamental, since it has minimal dimension  $\ell'$ .

#### 4.10. $\mathcal{U}_{pq}(\mathfrak{gl}_2)$

**4.10.1.** Let  $\mathcal{U}_{pq}(\mathfrak{gl}_2)$  be the associative unital  $\mathbb{C}(p, q)$ -algebra with generators

$$\{e, f, k^{\pm 1}, l^{\pm 1}\}$$

that satisfy the relations

$$\begin{aligned} k \cdot k^{-1} &= 1 = k^{-1}k, & l \cdot l^{-1} &= 1 = l^{-1} \cdot l, \\ k \cdot e \cdot k^{-1} &= p e, & k \cdot f \cdot k^{-1} &= p^{-1}f, \\ l \cdot e \cdot l^{-1} &= q e, & k \cdot f \cdot k^{-1} &= q^{-1}f, \\ e \cdot f - \frac{p}{q}f \cdot e &= \frac{k^2 - l^{-2}}{p - q^{-1}}, & k \cdot l &= l \cdot k. \end{aligned}$$

The algebra  $\mathcal{U}_{pq}(\mathfrak{gl}_2)$  is a Hopf algebra. The coproduct is

$$\begin{aligned} \Delta(k) &= k \otimes k, \\ \Delta(l) &= l \otimes l, \\ \Delta(e) &= e \otimes k + l^{-1} \otimes e, \\ \Delta(f) &= f \otimes k + l^{-1} \otimes f. \end{aligned}$$

Then the corresponding antipode map is

$$\begin{aligned} S(k) &= k^{-1}, \\ S(l) &= l^{-1}, \\ S(e) &= -qk^{-1}le, \\ S(f) &= -q^{-1}k^{-1}lf. \end{aligned}$$

The counit is given by

$$\begin{aligned} \epsilon(k) &= 1, & \epsilon(l) &= 1, \\ \epsilon(e) &= 0, & \epsilon(f) &= 0. \end{aligned}$$

**4.10.2.** In the special case  $p = q$ , define  $\mathcal{U}_q(\mathfrak{gl}_2) := \mathcal{U}_{pq}(\mathfrak{gl}_2)/\langle p - q \rangle$ , which is a 1-parameter deformed quantum enveloping algebra of  $\mathfrak{gl}_2$ . The two invertible generators  $\{k, l\}$  in its Cartan subalgebra obey the same relations. Therefore the elements  $\{kl^{-1}, k^{-1}l\}$  are central in  $\mathcal{U}_q(\mathfrak{gl}_2)$ .

Let  $\mathcal{U}_P(\mathfrak{sl}_2)$  be the *simply connected quantum group* of  $\mathfrak{sl}_2$  in the sense of [DCKP92, 0.3].  $\mathcal{U}_P(\mathfrak{sl}_2)$  is a  $\mathbb{C}(q)$ -algebra with generators  $\{e, f, l^{\pm 1}\}$  satisfying the relations

$$\begin{aligned} l \cdot e \cdot l^{-1} &= q e, & l \cdot f \cdot l^{-1} &= q^{-1} f, \\ e \cdot f - f \cdot e &= \frac{l^2 - l^{-2}}{q - q^{-1}}, & l \cdot l^{-1} &= 1 = l^{-1} \cdot l. \end{aligned}$$

$\mathcal{U}_P(\mathfrak{sl}_2)$  is a quotient algebra of  $\mathcal{U}_{pq}(\mathfrak{gl}_2)$ :

$$\mathcal{U}_P(\mathfrak{sl}_2) \simeq \mathcal{U}_{pq}(\mathfrak{gl}_2)/\langle p - q, k^{\pm 1} - l^{\pm 1} \rangle.$$



## Part II

*q*-oscillators



## CHAPTER 5

### $q$ -oscillators

#### 5.1. Introduction

In this chapter I discuss  $q$ -oscillator algebras (certain deformations of the quantum mechanical oscillator algebra) which were brought into the theory of quantum groups independently by Macfarlane, Hayashi and Biedenharn in 1989. The  $q$ -oscillator algebras that I discuss are not Hopf algebras. Nevertheless they have many intriguing properties reminiscent of the finite simple quantum group  $\mathcal{U}_q(\mathfrak{sl}_2)$  discussed in the last chapter.

In analogy with the quantum mechanical oscillator there is a natural Fock module representation. Remarkably, for the  $q$ -oscillator that I consider, the Fock module is essentially unique: in particular the vacuum vector is fixed to have “particle number” zero.

At a root of unity, the  $q$ -oscillator algebra has an enlarged centre and it is possible to construct finite dimensional irreducible representations — something which has no classical analogue: there exist cyclic, semicyclic [SG91] and nilpotent representations.

The infinite dimensional Fock module (at generic specialisation) and its irreducible quotient (at an even root of unity) are unitarisable.

The quantum enveloping algebra  $\mathcal{U}_q(\mathfrak{sl}_2)$  was first bosonised with a pair of  $q$ -oscillators by Macfarlane and Biedenharn. At about the same time Hayashi gave a bosonisation of each quantum enveloping algebra corresponding to the Lie algebras of simple type  $A$  and  $C$ , and of affine type  $A^{(1)}$  (and also  $q$ -spinor realisations of simple Lie algebras of type  $A$ ,  $B$  and  $D$ ). The bosonisation of  $\mathcal{U}_q(\mathfrak{g})$  can be used to construct representations of  $\mathcal{U}_q(\mathfrak{g})$  on a tensor product of  $q$ -oscillator representations.

I end this chapter with a short description of a 2-parameter deformation of the oscillator algebra, a quotient of which degenerates to the  $q$ -oscillator that I consider.

#### 5.2. Definitions

I begin by recalling the definition of the quantum mechanical oscillator algebra.

DEFINITION 5.2.1. The *oscillator algebra*  $\mathfrak{h}_4$  (Heisenberg-Weyl algebra) is a non-semisimple Lie algebra with generators  $\{\bar{n}, \bar{a}_+, \bar{a}_-, \bar{e}\}$  that satisfy the following relations:

$$\begin{aligned} [\bar{n}, \bar{a}_+] &= \bar{a}_+, & [\bar{n}, \bar{a}_-] &= -\bar{a}_-, \\ [\bar{a}_-, \bar{a}_+] &= \bar{e}, & \bar{e} &\text{ is central.} \end{aligned}$$

Since  $\mathfrak{h}_4$  is a Lie algebra, its universal enveloping algebra  $\mathcal{U}(\mathfrak{h}_4)$  is a Hopf algebra with the usual maps

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x, \\ \epsilon(x) &= 0, & (x \in \{\bar{a}_+, \bar{a}_-, \bar{n}, \bar{e}\}). \\ S(x) &= -x, \end{aligned}$$

For the unit element the Hopf maps are as usual:  $\Delta(1) = 1 \otimes 1$ ,  $\epsilon(1) = 1$  and  $S(1) = 1$ .

**5.2.2.** Usually physicists work with the quotient algebra  $\mathcal{U}'(\mathfrak{h}_4) := \mathcal{U}(\mathfrak{h}_4)/\langle \bar{e} - 1 \rangle$ . In  $\mathcal{U}'(\mathfrak{h}_4)$  the Heisenberg relation is then  $[\bar{a}_-, \bar{a}_+] = 1$ .

The definition of  $\mathfrak{h}_4$  has been made in terms of the step-up and step-down operators, which appear also to be more important when the generalisation to the  $q$ -oscillator is made.

DEFINITION 5.2.3. (compare [Mac89, Bie89, Hay90]) The  $q$ -oscillator algebra  $\mathcal{U}'_q(\mathfrak{h}_4)$  ( $q$ -Heisenberg-Weyl algebra) is the associative unital  $\mathbb{C}(q)$ -algebra with generators  $\{a_+, a_-, w, w^{-1}\}$  with the relations:

$$\begin{aligned} w \cdot w^{-1} &= 1 = w^{-1} \cdot w, \\ w \cdot a_+ \cdot w^{-1} &= q a_+, & a_- \cdot a_+ - q a_+ \cdot a_- &= w^{-1}, \\ w \cdot a_- \cdot w^{-1} &= q^{-1} a_-, & a_- \cdot a_+ - q^{-1} a_+ \cdot a_- &= w. \end{aligned}$$

**5.2.4.** There are two  $\mathbb{C}$ -algebra anti-automorphisms  $\omega_c$  and  $\omega_r$  of  $\mathcal{U}'_q(\mathfrak{h}_4)$  given by

$$\begin{aligned} \omega_c(w) &:= w^{-1}, & \omega_c(a_+) &:= a_-, & \omega_c(a_-) &:= a_+, & \omega_c(q) &:= q^{-1}, \\ \omega_r(w) &:= w, & \omega_r(a_+) &:= a_-, & \omega_r(a_-) &:= a_+, & \omega_r(q) &:= q. \end{aligned}$$

There is a  $\mathbb{C}$ -algebra automorphism  $\varphi$  of  $\mathcal{U}'_q(\mathfrak{h}_4)$  given by

$$\varphi(w) := w^{-1}, \quad \varphi(a_+) := a_+, \quad \varphi(a_-) := a_-, \quad \varphi(q) := q^{-1}.$$

REMARK 5.2.5. The algebra  $\mathcal{U}'_q(\mathfrak{h}_4)$  has two different  $q$ -Heisenberg algebra relations. It actually suffices to define deformations of the oscillator algebra with only one of these relations and this is done by many authors. But, as will be seen later in this chapter, both relations are needed in order to have an involutive anti-automorphism at a specialisation equal to a phase (necessary to construct a unitary representation of the  $q$ -oscillator algebra at  $q$  specialised to an even root of unity) and also for the existence of the bosonisation homomorphisms  $\mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}'_q(\mathfrak{h}_4)^{\otimes n}$ .

**5.2.6.** The two  $q$ -Heisenberg relations lead to the following relations in  $\mathcal{U}'_q(\mathfrak{h}_4)$ :

$$\begin{aligned} a_+ \cdot a_- - \frac{w - w^{-1}}{q - q^{-1}} &= 0, \\ a_- \cdot a_+ - \frac{qw - q^{-1}w^{-1}}{q - q^{-1}} &= 0, \end{aligned}$$

which can be compared to the the central element  $\bar{a}_+ \cdot \bar{a}_- - \bar{n} \cdot \bar{e}$  in  $\mathcal{U}(\mathfrak{h}_4)$ , which acts as zero on the standard vacuum representation.

From this it follows that the  $q$ -oscillator algebra  $\mathcal{U}'_q(\mathfrak{h}_4)$  has the following basis as a vector space over  $\mathbb{C}(q)$

$$\mathcal{U}'_q(\mathfrak{h}_4) = \sum_{m_- \in \mathbb{Z}_{>0}, m_0 \in \mathbb{Z}} \mathbb{C}(q) a_-^{m_-} w^{m_0} + \sum_{m_0 \in \mathbb{Z}} \mathbb{C}(q) w^{m_0} + \sum_{m_+ \in \mathbb{Z}_{>0}, m_0 \in \mathbb{Z}} \mathbb{C}(q) w^{m_0} a_+^{m_+}.$$

**5.2.7.** Let  $\epsilon \in \mathbb{C}^\times$ . Let  $\mathcal{U}'_{\mathcal{A}}(\mathfrak{h}_4)$  be the  $\mathbb{C}[q, q^{-1}]$ -subalgebra of  $\mathcal{U}'_q(\mathfrak{h}_4)$  generated by  $a_-, a_+, w^{\pm 1}$  and  $n := \frac{w^2 - w^{-2}}{2(q - q^{-1})}$ . Note that  $n$  is fixed by the anti-automorphisms  $\omega_c$  and  $\omega_r$  and has the commutation relations:

$$\begin{aligned} n \cdot a_+ - a_+ \cdot n &= \frac{1}{2} a_+ (qw^2 + q^{-1}w^{-2}), \\ a_- \cdot n - n \cdot a_- &= \frac{1}{2} (qw^2 + q^{-1}w^{-2}) a_-. \end{aligned}$$

Note that the second relation can be obtained from the first by applying the anti-automorphism  $\omega_c$  (or  $\omega_r$ ). Define the specialisation  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$  of  $\mathcal{U}'_{\mathcal{A}}(\mathfrak{h}_4)$  at  $q = \epsilon$  to be  $\mathcal{U}'_\epsilon(\mathfrak{h}_4) := \mathcal{U}'_{\mathcal{A}}(\mathfrak{h}_4) / \langle q - \epsilon \rangle$ .

**PROPOSITION 5.2.8.** *The quotient algebra  $\mathcal{U}'_1(\mathfrak{h}_4) / \langle w - 1, w^{-1} - 1 \rangle$  is  $\mathbb{C}$ -algebra isomorphic to  $\mathcal{U}'(\mathfrak{h}_4)$ .*

**PROOF.** Clearly in  $\mathcal{U}'_1(\mathfrak{h}_4) / \langle w - 1, w^{-1} - 1 \rangle$  the two  $q$ -Heisenberg-Weyl relations degenerate into the usual Heisenberg-Weyl relation of  $\mathcal{U}'(\mathfrak{h}_4)$ :  $[a_-, a_+] = 1$ . From the commutation relation between  $n$  and  $a_\pm$ , it follows that in the quotient algebra  $[n, a_+] = a_+$  and  $[a_-, n] = a_-$ . The proposition is proved.  $\square$

### 5.3. Other $q$ -deformed oscillator algebras

**5.3.1.** Clearly having two Heisenberg-Weyl relations is more restrictive than just one. Consider the deformed oscillator algebra  $\mathcal{A}_4$  with generators  $\{A_+, A_-, W^{\pm 1}\}$  that satisfy the following relations:

$$\begin{aligned} W \cdot W^{-1} &= 1 = W^{-1} \cdot W, \\ W \cdot A_+ \cdot W^{-1} &= qA_+, & A_- \cdot A_+ - q^2 A_+ \cdot A_- &= 1, \\ W \cdot A_- \cdot W^{-1} &= q^{-1}A_-. \end{aligned}$$

Note that  $\mathcal{A}_4$  has only one  $q$ -Heisenberg-Weyl type relation.

LEMMA 5.3.2. *Let  $n \in \mathbb{Z}$ . The algebra  $\mathcal{A}_4$  is  $\mathbb{C}[q, q^{-1}]$ -algebra isomorphic to the algebra with the generators  $\{A'_+, A'_-, W'^{\pm 1}\}$  and the relations*

$$\begin{aligned} W' \cdot W'^{-1} &= 1 = W'^{-1} \cdot W', \\ W' \cdot A'_+ \cdot W'^{-1} &= qA'_+, & A'_- \cdot A'_+ - q^{2-n}A'_+ \cdot A'_- &= W'^{-n}, \\ W' \cdot A'_- \cdot W'^{-1} &= q^{-1}A'_-. \end{aligned}$$

PROOF. The isomorphism is given by the following map

$$W \mapsto W', \quad A_- \mapsto W'^n A'_-, \quad A_+ \mapsto A'_+.$$

□

**5.3.3.** Celeghini, Giachetti, Sorace and Tarlini [CGST91] introduced a  $\mathbb{C}(q)$ -algebra  $\mathcal{U}_q(\mathfrak{h}_4)$  with generators  $\{b_+, b_-, m, c, c^{-1}\}$  that satisfy

$$\begin{aligned} c \text{ is central,} & & b_- \cdot b_+ - b_+ \cdot b_- &= \frac{c - c^{-1}}{q - q^{-1}}, \\ c \cdot c^{-1} = 1 = c^{-1} \cdot c, & & [m, b_{\pm}] &= \pm b_{\pm}. \end{aligned}$$

The algebra is a Hopf algebra:

$$\begin{aligned} \Delta(b_+) &= b_+ \otimes 1 + c \otimes b_+, & S(b_+) &= -c^{-1}b_+, \\ \Delta(b_-) &= b_- \otimes c^{-1} + 1 \otimes b_-, & S(b_-) &= -cb_-, \\ \Delta(c) &= c \otimes c, & S(c) &= c^{-1}, \\ \Delta(m) &= m \otimes 1 + 1 \otimes m, & S(m) &= -m, \\ \epsilon(b_+) &= 0, & \epsilon(b_-) &= 0, \\ \epsilon(c) &= 1, & \epsilon(m) &= 0. \end{aligned}$$

This algebra has a nontrivial Hopf algebra structure and it is known [GS91] to be related to link invariants.

LEMMA 5.3.4. *Let  $\mathcal{U}_A(\mathfrak{h}_4)$  be the  $\mathbb{C}[q, q^{-1}]$ -subalgebra of  $\mathcal{U}_q(\mathfrak{h}_4)$  generated by*

$$\{b_+, b_-, m, c^{\pm}, e := \frac{c - c^{-1}}{q - q^{-1}}\}$$

*(so that  $[b_-, b_+] = e$ ). Let  $\epsilon \in \mathbb{C}^{\times}$  and define  $\mathcal{U}_{\epsilon}(\mathfrak{h}_4) := \mathcal{U}_A(\mathfrak{h}_4) / \langle q - \epsilon \rangle$ . The following map is a  $\mathbb{C}$ -algebra isomorphism  $\mathcal{U}_{\epsilon}(\mathfrak{h}_4) / \langle c^{\pm 1} - 1 \rangle \rightarrow \mathcal{U}(\mathfrak{h}_4)$*

$$\begin{aligned} b_+ &\mapsto \bar{a}_+, & b_- &\mapsto \bar{a}_-, \\ m &\mapsto \bar{n}, & e &\mapsto \bar{e}. \end{aligned}$$

From the lemma it can be deduced that the basic representations of  $\mathcal{U}_{\epsilon}(\mathfrak{h}_4)$  are equivalent (even at  $\epsilon$  a root of unity) to those of  $\mathcal{U}(\mathfrak{h}_4)$ . Therefore I will not discuss this algebra further in this chapter.

**5.3.5.** Schwenk and Wess have studied [SW92] another interesting deformation of the Heisenberg-Weyl algebra [Man91] with generators  $\{x, p\}$  and relation

$$p \cdot x - q x \cdot p = -i.$$

REMARK 5.3.6. When considering the Hopf algebra structure of  $\mathfrak{h}_4$  ( $\mathcal{U}_q(\mathfrak{h}_4)$ ), the generator  $\bar{e}$  ( $c$ ) plays an important role. The quotient algebra  $\mathcal{U}'(\mathfrak{h}_4)$  cannot enjoy a Hopf algebra structure because of the relation  $\bar{e} = 1$  (see for instance [Pal93]):

$$\begin{aligned} \Delta(\bar{e}) &= \bar{e} \otimes 1 + 1 \otimes \bar{e} \equiv 2(1 \otimes 1) \\ &\neq \Delta(1) = 1 \otimes 1. \end{aligned}$$

It follows that  $\mathcal{U}'_q(\mathfrak{h}_4)$  does not have a canonical coproduct (with a well defined classical limit).

#### 5.4. A Fock Representation of $\mathcal{U}'_q(\mathfrak{h}_4)$

LEMMA 5.4.1. *Let  $n \in \mathbb{Z}_{>0}$ . The follow identities hold in  $\mathcal{U}'_q(\mathfrak{h}_4)$ :*

$$\begin{aligned} a_- a_+^n &\equiv q^{\pm n} a_+^n a_- + [n]_q a_-^{n-1} w^{\mp 1}, \\ a_-^n a_+ &\equiv q^{\pm n} a_+ a_-^n + [n]_q a_+^{n-1} w^{\mp 1}. \end{aligned}$$

**5.4.2.** Let  $F$  be a  $\mathbb{C}(q)$ -vector space with basis  $\{u_n \mid n \in \mathbb{N}\}$

$$F := \sum_{n \in \mathbb{N}} \mathbb{C}(q) u_n.$$

The following action of  $\mathcal{U}'_q(\mathfrak{h}_4)$  on  $F$ , makes  $F$  a  $\mathcal{U}'_q(\mathfrak{h}_4)$ -module

$$\begin{aligned} a_+ \cdot u_n &:= u_{n+1}, \\ w \cdot u_n &:= q^n u_n, \\ a_- \cdot u_n &:= [n]_q u_{n-1}. \end{aligned}$$

For  $n \in \mathbb{Z}_{<0}$ , I set  $u_n = 0$ .

The vector  $u_0$  is a vacuum vector (lowest weight vector)

$$\begin{aligned} a_- \cdot u_0 &= 0, \\ w \cdot u_0 &= u_0, \\ F &= \mathcal{U}'_q(\mathfrak{h}_4) \cdot u_0. \end{aligned}$$

Note that the weight ( $w$ -eigenvalue) of  $u_0$  is essentially fixed (up to a factor of  $-1$ ) by the two  $q$ -Heisenberg-Weyl relations.

REMARK 5.4.3. Consider for a moment the construction of a Fock module  $\tilde{F}$  over the deformed oscillator algebra  $\mathcal{A}_4$  defined in 5.3.1. The weight of the vacuum vector  $v_0$  of  $\tilde{F}$  is not fixed by the single  $q$ -Heisenberg-Weyl relation. In fact there are many different vacuum states that can be defined:  $w \cdot v_0 = q^{n_0} v_0$  ( $n_0 \in \mathbb{Z}$ ).

PROPOSITION 5.4.4. *The  $\mathcal{U}'_q(\mathfrak{h}_4)$ -module  $F$  is irreducible.*

PROOF. There is not a (singular) vector  $u'$  in  $F$ , such that  $a_- \cdot u' = 0$ , except  $u' = u_0$ .  $\square$

### 5.5. The Centre of $\mathcal{U}'_q(\mathfrak{h}_4)$ at a root of unity

Let  $\ell$  be a positive integer, such that  $\ell > 2$ . Fix  $\epsilon$  to be a primitive  $\ell$ -th root of unity (then  $\epsilon^\ell = 1$ ).

NOTATION. Define  $\ell'$  to be

$$\ell' := \begin{cases} \ell & \text{if } \ell \text{ is odd,} \\ \frac{\ell}{2} & \text{if } \ell \text{ is even.} \end{cases}$$

PROPOSITION 5.5.1. *At the root of unity  $\epsilon$  the following elements lie in the centre of  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$*

$$\{a_+^\ell, a_-^\ell, w^\ell, w^{-\ell}\}.$$

PROOF. The proof of the proposition is straightforward using the defining relations of  $\mathcal{U}'_q(\mathfrak{h}_4)$  and lemma 5.4.1.  $\square$

At the root of unity  $\epsilon$  the algebra  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$  is finite dimensional (finitely generated as a free module) over its centre.

### 5.6. The Fock module at a root of unity

Let  $F_{\mathcal{A}}$  be the  $\mathcal{U}'_{\mathcal{A}}(\mathfrak{h}_4)$ -submodule of  $F$  over  $\mathbb{C}[q, q^{-1}]$ . Define the specialisation  $F_\epsilon$  of the  $\mathcal{U}'_{\mathcal{A}}(\mathfrak{h}_4)$ -module  $F_{\mathcal{A}}$  at  $q = \epsilon$ , to be the  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -submodule of  $F_{\mathcal{A}}$  over  $\mathbb{C}$ .

I consider now the Fock module  $F_\epsilon$  over  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$  at the root of unity  $\epsilon$ .

PROPOSITION 5.6.1. *The  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module  $F_\epsilon$  at the root of unity  $\epsilon$  is reducible and has a finite number of weight spaces.*

PROOF. The elements in  $\{u_{k\ell'} \mid k \in \mathbb{Z}_{>0}\}$  are all singular vectors in  $F_\epsilon$ , since

$$a_- \cdot u_{k\ell'} = [k\ell']_\epsilon u_{k\ell'-1} \equiv 0.$$

Therefore  $F_\epsilon$  is reducible. Consider now the action of  $w$  on  $F_\epsilon$

$$w \cdot u_n = \epsilon^n u_n.$$

The possible weights of  $w$  are then  $\{1, \epsilon, \epsilon^2, \dots, \epsilon^{\ell-1}\}$ . So there are only  $\ell$  weight spaces:  $F_\epsilon = \bigoplus_{n=0}^{\ell-1} F_{(n)}$  ( $F_{(n)} := \{v \in F_\epsilon \mid w \cdot v = \epsilon^n v\}$ ).  $\square$

**5.6.2.** Let  $k \in \mathbb{N}$ . Denote by  $F'_{k\ell'}$  the submodule generated by the singular vector  $u_{k\ell'}$ . Note that  $F'_0 \equiv F_\epsilon$ . Denote by  $L_\epsilon$  the irreducible quotient module  $F_\epsilon / F'_{\ell'}$ .

LEMMA 5.6.3. *Let  $m, n \in \mathbb{N}$ , such that  $m < n$ .*

- (1) *The quotient module  $F'_{m\ell'} / F'_{n\ell'}$  is irreducible if and only if  $n = m + 1$ .*
- (2) *The quotient module  $F'_{m\ell'} / F'_{(m+1)\ell'}$  is equivalent to  $L_\epsilon$ .*

PROOF. (1) If  $n > m + 1$  then clearly the quotient module contains (the image of) the singular vector  $u_{(m+1)\ell'}$ , so it is reducible. On the other hand if  $n = m + 1$  then  $a_+$  and  $a_-$  act transitively and the module is irreducible and has dimension  $\ell'$ . (2) is obvious since  $[m + k\ell']_\epsilon = (-1)^{\frac{k\ell}{\ell'}} [m]_\epsilon$ .  $\square$



### 5.7. Semicyclic Representations

**5.7.1.** Let  $(\lambda, \mu) \in \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}$ . Let  $V_{\lambda, \mu} := \sum_{n=0}^{\ell-1} \mathbb{C}v_n$  be a  $\mathbb{C}$ -vector space. At the root of unity  $\epsilon$ , define the following  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module structure on  $V_{\lambda, \mu}$

$$\begin{aligned} a_+ \cdot v_n &:= \begin{cases} v_{n+1} & \text{if } n \in [0, \ell-2] \\ \lambda v_0 & \text{if } n = \ell-1 \end{cases} \\ a_- \cdot v_n &:= \begin{cases} [n]_\epsilon v_{n-1} & \text{if } n \in [1, \ell-1] \\ \mu v_{\ell-1} & \text{if } n = 0 \end{cases} \\ w \cdot v_n &:= \epsilon^n v_n. \end{aligned}$$

Note that  $V_{0,0}$  is equivalent to the quotient  $F_\epsilon/F'_\ell$ .

LEMMA 5.7.2 (ODD CASE). *Let  $\ell$  be odd.*

- (1)  $V_{\lambda, \mu}$  is an irreducible  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module.
- (2) If  $(\lambda, \mu) \in \mathbb{C}^\times \times \{0\} \cup \{0\} \times \mathbb{C}^\times$ , then  $V_{\lambda, \mu}$  is a semicyclic (semiperiodic)  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module.
- (3)  $V_{0,0}$  is a nilpotent  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module.

PROOF. (1) At an odd root of unity,  $V_{\lambda, \mu}$  does not contain any  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -submodules. Therefore it is irreducible. (2) If  $\lambda$  (respectively  $\mu$ ) is zero, then  $a_+$  ( $a_-$ ) acts nilpotently and  $a_-$  ( $a_+$ ) injectively. So the module is semicyclic:  $a_+^\ell \cdot v_n = \lambda v_n$  and  $a_-^\ell \cdot v_n = \mu [\ell-1]_\epsilon! v_n$  ( $n \in [0, \ell-1]$ ). (3) From  $(\lambda, \mu) = (0, 0)$ , it is clear that  $V_{0,0}$  is nilpotent.  $\square$

LEMMA 5.7.3 (EVEN CASE). *Let  $\ell$  be even.*

- (1)  $a_-$  acts nilpotently in  $V_{\lambda, \mu}$ .
- (2)  $V_{0,0}$  is a reducible nilpotent  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module. The irreducible quotient of  $V_{0,0}$  is equivalent as a  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module to  $L_\epsilon$ .
- (3) Let  $(\lambda, \mu) \in \{0\} \times \mathbb{C}^\times$ .  $V_{0, \mu}$  is an irreducible nilpotent  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module.
- (4) Let  $(\lambda, \mu) \in \mathbb{C}^\times \times \{0\}$ .  $V_{\lambda, 0}$  is an irreducible semicyclic  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module.

PROOF. (1) For even  $\ell$ ,  $[\ell']_\epsilon = 0$  and  $a_-$  acts nilpotently, annihilating the vector  $v_{\ell'}$ . (2)  $(\lambda, \mu) = (0, 0)$ , so  $V_{0,0}$  is nilpotent.  $V_{0,0}$  contains a submodule generated by  $v_{\ell'}$ . Hence it is reducible. (3)  $a_+$  acts nilpotently since  $\lambda = 0$  and  $a_-$  acts nilpotently by (1).  $V_{0, \mu}$  is generated by  $v_0$  and contains a submodule generated by  $v_{\ell'}$ . (4) follows since  $a_+$  acts cyclically, whereas  $a_-$  acts nilpotently by (1).  $\square$

REMARK 5.7.4. It can be checked that the space of the parameters  $\lambda$  and  $\mu$  of the module  $V_{\lambda, \mu}$  cannot be extended to  $\mathbb{C} \times \mathbb{C}$ , since an action of  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$  on  $V_{\mu, \nu}$  with  $(\lambda, \mu) \in \mathbb{C}^\times \times \mathbb{C}^\times$  does not form a consistent (cyclic) representation.

### 5.8. Cyclic Representations

**PROPOSITION 5.8.1.** *Let  $\ell$  be even. There does not exist an  $\ell$  (or  $\ell'$ ) dimensional (minimal) cyclic (periodic) representation of  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ .*

**PROOF.** There is no  $\ell'$ -dimensional cyclic representation, since  $a_+^{\ell'}$  and  $a_-^{\ell'}$  are not central in  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ . In fact from 5.4.1 it is clear that  $a_\pm^{\ell'}$  anti-commutes with  $a_\mp$ ,  $w$  and  $w^{-1}$ . So it is not possible to have a cyclic representation in this case. There is no  $\ell$ -dimensional cyclic representation, since by 5.7.3(1) the action of  $a_-$  would be nilpotent.  $\square$

**PROPOSITION 5.8.2.** *Let  $(\lambda, \mu, \zeta) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^\times$ . Consider the left quotient*

$$M_{\lambda, \mu, \zeta} := \mathcal{U}'_\epsilon(\mathfrak{h}_4) / \langle a_+^\ell - \lambda, a_-^\ell - \mu, w - \zeta \rangle_L.$$

*It is a  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module of dimension  $2\ell - 1$ . Let  $v$  denote the image of 1 in  $M_{\lambda, \mu, \zeta}$ . For odd  $\ell$  and  $(\lambda, \mu) \in \mathbb{C}^\times \times \mathbb{C}^\times$ , it is an irreducible cyclic (periodic)  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module. For  $\ell$  even, if  $\zeta \notin \{1, -1\}$  then  $M_{\lambda, \mu, \zeta}$  is an irreducible cyclic  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module,*

**PROOF.** The dimension of the module follows from 5.2.6. For odd  $\ell$ , the module is cyclic and irreducible, since by 5.4.1  $a_+$  and  $a_-$  act injectively. For even  $\ell$ ,  $a_+$  and  $a_-$  also act injectively, unless  $\zeta \in \{1, -1\}$  in which case the module is nilpotent and contains a submodule generated by  $a_+^{\ell'}v$ .  $\square$

**REMARK 5.8.3.** Most of the statements that have been made on the centre and representations of  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$  at the root of unity  $\epsilon$  are also true for the algebra  $\mathcal{A}_4$  at the root of unity  $\epsilon$ : the elements  $\{A_+^\ell, A_-^\ell, W^\ell, W^{-\ell}\}$  are central, there exists a similar set of representations and so on (see also the following chapter on this point).

### 5.9. Unitary Representations

**5.9.1.** Let  $F$  be the  $\mathbb{C}(q)$ -Fock module of  $\mathcal{U}'_q(\mathfrak{h}_4)$  defined in 5.4. Let  $\omega$  be one of the involutive ( $\omega^2 = \text{id}$ )  $\mathbb{C}(q)$ -algebra anti-automorphisms of  $\mathcal{U}'_q(\mathfrak{h}_4)$  defined in 5.2.4 and require that  $\omega(\alpha) := \alpha^*$  ( $\alpha \in \mathbb{C}$ ). Define on  $F$  a scalar product  $(\cdot, \cdot) : \mathcal{U}'_q(\mathfrak{h}_4) \times \mathcal{U}'_q(\mathfrak{h}_4) \rightarrow \mathbb{C}(q)$ , contravariant with respect to  $\omega$ , such that

$$\begin{aligned} (u_m, u_n) &:= \delta_{m,n} [m]_q! & (m, n \in \mathbb{N}), \\ (x \cdot v, w) &= (v, \omega(x) \cdot w) & (x \in \mathcal{U}'_q(\mathfrak{h}_4); v, w \in F). \end{aligned}$$

Denote also by  $(\cdot, \cdot)$  the corresponding induced scalar products on  $F_{\mathcal{A}}$  and  $F_\epsilon$ .

**LEMMA 5.9.2.** *The triple  $(F, (\cdot, \cdot), \omega)$  is a  $\star$ -representation of  $\mathcal{U}'_q(\mathfrak{h}_4)$ .*

**PROPOSITION 5.9.3.** *Let  $\omega_r$  be the anti-automorphism of  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$  defined in 5.2.4 (with  $\omega_r(\alpha) := \alpha^*$  ( $\alpha \in \mathbb{C}$ )). If  $\epsilon \in \mathbb{R}_{>0}$ , then the  $\star$ -representation  $(F_\epsilon, (\cdot, \cdot), \omega_r)$  of  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$  is unitary.*

**PROOF.** When  $\epsilon$  is real and positive, then  $[n]_\epsilon \in \mathbb{R}_{>0}$  (for all  $n \in \mathbb{Z}_{>0}$ ). Therefore in this case the scalar product is positive definite and the  $\star$ -representation is unitary.  $\square$

LEMMA 5.9.4. *Let  $\epsilon \in \mathbb{C}^\times$  be such that  $|\epsilon| = 1$ . Then the triple  $(F_\epsilon, (\cdot, \cdot), \omega_c)$  is a  $\star$ -representation of  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ .*

PROPOSITION 5.9.5. *Let  $\ell$  be an even positive integer ( $\ell \geq 4$ ). Let  $\ell' := \frac{\ell}{2}$  and let  $\epsilon$  be a primitive  $\ell$ -th root of unity. Consider the irreducible quotient  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ -module  $L_\epsilon$  at the root of unity  $\epsilon$ , defined in 5.6.2 and denote again by  $(\cdot, \cdot) : L_\epsilon \times L_\epsilon \rightarrow \mathbb{C}$  the induced scalar product on  $L_\epsilon$ . The  $\star$ -representation  $(L_\epsilon, (\cdot, \cdot), \omega_c)$  is a unitary representation of  $\mathcal{U}'_\epsilon(\mathfrak{h}_4)$ .*

PROOF. The scalar product on  $L_\epsilon$  gives  $(u_m, u_m) = [m]_\epsilon!$ . Since  $[m]_\epsilon = \frac{\sin(\frac{m\pi i}{\ell'})}{\sin(\frac{i\pi}{\ell'})}$ ,  $[m]_\epsilon > 0$  for  $m \in [1, \ell' - 1]$ . Therefore  $[m]_\epsilon! > 0$  ( $m \in [1, \ell' - 1]$ ) and the scalar product on  $L_\epsilon$  is positive definite.  $\square$

REMARK 5.9.6. There does not exist a unitary representation at an odd root of unity, because  $[m]_\epsilon < 0$  for  $m \in [\frac{\ell+1}{2}, \ell - 1]$ , so the scalar product is not positive definite.

### 5.10. Bosonisation of $\mathcal{U}_q(\mathfrak{sl}_n)$

The simple quantum groups corresponding to Cartan matrices of type  $A_n$  and  $C_n$  can be bosonised [Hay90] with  $\mathcal{U}'_q(\mathfrak{h}_4)$ . Here I present only the bosonisation of  $\mathcal{U}_q(\mathfrak{sl}_n)$ .

NOTATION. For  $x, y \in \mathcal{U}'_q(\mathfrak{h}_4)$ , define

$$x \overset{i}{\otimes} y := \underbrace{1 \otimes \cdots \otimes 1}_{i-1 \text{ times}} \otimes x \otimes y \otimes 1 \cdots 1 \in \mathcal{U}'_q(\mathfrak{h}_4)^{\otimes n}.$$

THEOREM 5.10.1. [Hay90, 3.2] *The following map  $\mathcal{U}_q(\mathfrak{sl}_n) \rightarrow \mathcal{U}'_q(\mathfrak{h}_4)^{\otimes n}$  is an  $\mathbb{C}(q)$ -algebra homomorphism*

$$\begin{aligned} e_i &\mapsto a_- \overset{i}{\otimes} a_+, \\ f_i &\mapsto a_+ \overset{i}{\otimes} a_-, \\ k_i &\mapsto w^{-1} \overset{i}{\otimes} w. \end{aligned}$$

PROOF. The homomorphism is easily verified on the relations  $k_i e_j k_i^{-1} = q^{a_{ij}} e_j$  and  $k_i f_j k_i^{-1} = q^{-a_{ij}} f_j$ . Similarly the map is checked on the relation  $[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}$  using the relations in 5.2.6. The Serre relations require a little more work.  $\square$

**5.10.2.** Using the theorem, an infinite dimension (unitary) representation of  $\mathcal{U}_q(\mathfrak{sl}_n)$  can be constructed on  $F^{\otimes n}$ . At the root of unity  $\epsilon$ , a finite dimensional (unitary) representation of  $\mathcal{U}_\epsilon(\mathfrak{sl}_n)$  can be constructed on  $L_\epsilon^{\otimes n}$  and semicyclic (cyclic) representations of  $\mathcal{U}_\epsilon(\mathfrak{sl}_n)$  can be constructed on  $V_{\lambda, \mu}^{\otimes n}$  ( $M_{\lambda, \mu}^{\otimes n}$  respectively). Interestingly a mixture of different representations can also be taken.

**5.11. 2-parameter deformed  $pq$ -oscillator**

Consider the associative unital  $\mathbb{C}(p, q)$ -algebra  $\mathcal{U}'_{pq}(\mathfrak{h}_4)$  with generators

$$\{a_+, a_-, w^{\pm 1}, x^{\pm 1}\},$$

that satisfy the relations

$$\begin{aligned} w \cdot w^{-1} &= 1 = w^{-1} \cdot w, & x \cdot x^{-1} &= 1 = x^{-1} \cdot x, \\ w \cdot a_+ \cdot w^{-1} &= pa_+, & w \cdot a_- \cdot w^{-1} &= p^{-1}a_-, \\ x \cdot a_+ \cdot x^{-1} &= qa_+, & x \cdot a_- \cdot x^{-1} &= q^{-1}a_-, \\ a_- \cdot a_+ - pa_+ \cdot a_- &= x^{-1}, & a_- \cdot a_+ - q^{-1}a_+ \cdot a_- &= w. \end{aligned}$$

It is a two parameter deformation of  $\mathcal{U}'(\mathfrak{h}_4)$ .

LEMMA 5.11.1. *The following map  $\mathcal{U}'_{pq}(\mathfrak{h}_4) \rightarrow \mathcal{U}'_q(\mathfrak{h}_4)$  is an anti-automorphism of  $\mathcal{U}'_{pq}(\mathfrak{h}_4)$*

$$\begin{aligned} a_+ &\mapsto a_-, & a_- &\mapsto a_+, & w &\mapsto x^{-1}, \\ x &\mapsto w^{-1}, & p &\mapsto q^{-1}, & q &\mapsto p^{-1}. \end{aligned}$$

**5.11.2.** The quotient algebra  $\mathcal{U}'_{pq}(\mathfrak{h}_4)/\langle p - q, w - x \rangle$  is isomorphic to  $\mathcal{U}'_q(\mathfrak{h}_4)$ .

## CHAPTER 6

# A Quadratic 2-parameter deformation of the Oscillator Algebra

### 6.1. Introduction

In this chapter I consider a new 2-parameter deformation [Pet93] of the oscillator algebra  $\mathcal{U}(\mathfrak{h}_4)$ , with a natural  $q$ -Heisenberg-Weyl subalgebra.

**6.1.1. Quadratic deformations.** In the theory of (classical) differential geometry of a Lie group  $G$ , its Lie algebra  $\mathfrak{g}$  appears naturally as the set of left invariant vector fields  $\mathcal{L}(G)$  on  $G$  with a commutator operation. There is an elegant theory of noncommutative differential geometry on a matrix quantum group  $\text{fun}_q(G)$  (see [Wor89, Man89, WZ90]), from which it turns out that the quantum Lie algebra that arises from the set of invariant vector fields on  $\text{fun}_q(G)$ , is not the quantum enveloping algebra  $\mathcal{U}_q(\mathfrak{g})$ , but rather a quadratic Hopf algebra  $\mathcal{L}_q(G)$ . The quantum calculus of the simplest case, the compact quantum group  $\text{fun}_q(SU_2)$ , was first studied by Woronowicz [Wor87b] and he found the quadratic quantum Lie algebra  $\mathcal{L}_q(SU_2)$ . (Sklyanin had already introduced a quadratic deformation of  $\mathcal{U}(\mathfrak{sl}_2)$  in [Skl85]: though it was not a Hopf algebra.) Later a deformation of  $\mathcal{U}(\mathfrak{sl}_2)$  similar to the one found by Woronowicz was uncovered [Wit90] in the context of vertex models. Fairlie [Fai90] and Curtright and Zachos [CZ90] generalised this algebra to a 2-parameter deformation  $\mathcal{U}_{qr}(\mathfrak{sl}_2)$  of  $\mathcal{U}(\mathfrak{sl}_2)$ .

**6.1.2.** An interesting problem is the study of twisted deformations of multidimensional oscillator algebras. Deformed multi-oscillators, covariant under the action of a quantum group, have been studied by Pusz and Woronowicz [PW89, Pus89] and Zumino [Zum91]. Twisted multiparameter deformed multi-oscillators have been studied in [FZ91, FN93]. Another challenge is the construction of  $q$ -quantum mechanical systems [SW92, CC91].

**6.1.3.** I describe the contents of this chapter. Following an idea I got from the 2-parameter quadratic deformation  $\mathcal{U}_{qr}(\mathfrak{sl}_2)$  of  $\mathcal{U}(\mathfrak{sl}_2)$  that I mentioned above, I introduce a 2-parameter deformation  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  of the oscillator algebra  $\mathcal{U}(\mathfrak{h}_4)$  and its one parameter deformed Heisenberg-Weyl subalgebra  $\mathcal{U}_r(\mathfrak{h}_3)$ . It has a family of Fock modules parametrised by the  $q$ -number operator eigenvalue of the vacuum vector and the

central charge of the central element. It is possible to unitarise these Fock modules over  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  when the parameters are specialised to positive real numbers.

At an  $\ell$ -th root of unity the specialisation  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$  of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  has an enlarged centre: the step-up and step-down operators to the  $\ell'$ -th power are in the centre of  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$ . There is a finite dimensional irreducible quotient of the Fock module, which may have some connection with paragrassmann algebras [FIK92]. There also exist semicyclic and cyclic modules.

One of the important applications of  $q$ -oscillators, mentioned in the last chapter, is the bosonisation of other quadratic algebras and quantum groups. The  $q$ -oscillator discussed in the previous chapter was ideally suited for bosonising quantum enveloping algebras. The 1-parameter quadratic deformation of  $\mathcal{U}(\mathfrak{sl}_2)$  can be bosonised by the quadratic  $r$ -Heisenberg algebra  $\mathcal{U}_r(\mathfrak{h}_3)$ . It is also possible to bosonise the centreless  $q$ -Virasoro ( $q$ -Witt) algebra with an extension of  $\mathcal{U}_r(\mathfrak{h}_3)$ . From this realisation at a root of unity, some finite dimensional representations of the  $q$ -Witt algebra can be constructed (compare [NQ91]). Kassel has found [Kas92] a full  $q$ -analogue of the Virasoro algebra with central extension.

In [CGST90] Wigner-Iönu contractions of quantum groups were first performed. The quantum group  $\mathcal{U}_q(\mathfrak{sl}_2)$  (defined in 4.9) was contracted to Heisenberg and Euclidean quantum groups. A contraction of  $\mathcal{U}_{qr}(\mathfrak{sl}_2)$ , at the specialisation  $q = 1$ , leads to  $\mathcal{U}_r(\mathfrak{h}_3)$ . Another contraction gives a 2-parameter quadratic deformed Euclidean algebra.

The quadratic algebra  $\mathcal{U}_r(\mathfrak{h}_3)$  is covariant under left and right coactions of  $\text{fun}_{r^2}(SL_2)$ .

## 6.2. A Quadratic deformation of $\mathfrak{sl}_2$

NOTATION. Define the  $q$ -commutator  $[X, Y]_q := qX \cdot Y - q^{-1}Y \cdot X$ .

DEFINITION 6.2.1. [Fai90, CZ90] Define  $\mathcal{U}_{qr}(\mathfrak{sl}_2)$  to be the associative unital algebra over  $\mathbb{C}[q, q^{-1}, r, r^{-1}]$  with generators  $\{W_+, W_-, W_0\}$  that satisfy the following relations

$$\begin{aligned} [W_0, W_+]_q &= W_+, \\ [W_-, W_0]_q &= W_-, \\ [W_+, W_-]_{r^{-1}} &= W_0. \end{aligned}$$

**6.2.2.** The algebra  $\mathcal{U}_{qr}(\mathfrak{sl}_2)$  is an example of what is called a quadratic algebra: it has defining relations that are quadratic in its generators. It seems that  $\mathcal{U}_{qr}(\mathfrak{sl}_2)$  is only a Hopf algebra for certain values of  $q$  and  $r$ : specifically when  $q = r^2$  or  $r = q^2$  (the latter corresponding to the quantum algebra of [Wit90]). This agrees with results in the literature that the number of deformation parameters of a Hopf algebra deformation of  $\mathcal{U}(\mathfrak{g})$  ( $\mathfrak{g}$  a simple Lie algebra of rank  $r$ ) is bounded by  $r$ .

## 6.3. Definition of $\mathcal{U}_{qr}(\mathfrak{h}_4)$

**6.3.1.** Denote by  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  the associative unital  $\mathbb{C}[q, q^{-1}, r, r^{-1}]$ -algebra with generators  $\{A_+, A_-, N, E\}$  that satisfy the relations

$$\begin{aligned} E &\text{ is central,} \\ [N, A_+]_q &= A_+, \\ [A_-, N]_q &= A_-, \\ [A_-, A_+]_r &= E. \end{aligned}$$

It is easy to check that the quantum algebra  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  is well defined by these relations. In particular it does not matter in which way a cubic monomial  $X_1 X_2 X_3$  in the generators of the quantum algebra is re-ordered: the result should be the same answer either way. For example in  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  the re-orderings

$$\begin{aligned} NA_+A_- &= r^2 NA_-A_+ - rEN \\ &= q^2 r^2 A_-NA_+ - qr^2 A_-A_+ - rEN \\ &= r^2 A_-A_+N - rEN \end{aligned}$$

and

$$\begin{aligned} NA_+A_- &= q^{-2} A_+NA_- + q^{-1} A_+A_- \\ &= A_+A_-N \\ &= r^2 A_-A_+N - rEN \end{aligned}$$

give the same expression.

**6.3.2.** Let  $\epsilon, \zeta \in \mathbb{C}^\times$ . Define the partial specialisation  $\mathcal{U}_{q\zeta}(\mathfrak{h}_4)$  of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  at  $r = \zeta$  to be the  $\mathbb{C}[q, q^{-1}]$ -algebra  $\mathcal{U}_{q\zeta}(\mathfrak{h}_4) := \mathcal{U}_{qr}(\mathfrak{h}_4) / \langle r - \zeta \rangle$ . The full specialisation  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$  of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  at  $(q, r) = (\epsilon, \zeta)$  is defined to be the  $\mathbb{C}$ -algebra  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4) := \mathcal{U}_{qr}(\mathfrak{h}_4) / \langle q - \epsilon, r - \zeta \rangle$ .

**LEMMA 6.3.3.** *The specialisation  $\mathcal{U}_{1,1}(\mathfrak{h}_4)$  of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  is isomorphic to the universal enveloping algebra  $\mathcal{U}(\mathfrak{h}_4)$  defined in 5.2.1. The quotient  $\mathcal{U}_{1,1}(\mathfrak{h}_4) / \langle E - 1 \rangle$  is isomorphic to  $\mathcal{U}'(\mathfrak{h}_4)$ .*

**6.3.4.** The algebra  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  has a subalgebra  $\mathcal{U}_r(\mathfrak{h}_3)$  generated by  $\{A_+, A_-, E\}$ . I call it the  $r$ -Heisenberg-Weyl subalgebra of the  $qr$ -oscillator algebra  $\mathcal{U}_{qr}(\mathfrak{h}_4)$ . Let  $\zeta \in \mathbb{C}^\times$ . I define the specialisation  $\mathcal{U}_\zeta(\mathfrak{h}_3)$  of  $\mathcal{U}_r(\mathfrak{h}_3)$  at  $r = \zeta$  to be  $\mathcal{U}_\zeta(\mathfrak{h}_3) := \mathcal{U}_r(\mathfrak{h}_3) / \langle r - \zeta \rangle$ . The specialisation at  $r = 1$  is isomorphic to the subalgebra  $\mathcal{U}(\mathfrak{h}_3)$  of  $\mathcal{U}(\mathfrak{h}_4)$  generated by  $\{\bar{a}_+, \bar{a}_-, \bar{e}\}$ .

**LEMMA 6.3.5.** *Let  $\mathcal{U}'_r(\mathfrak{h}_3) := \mathcal{U}_r(\mathfrak{h}_3) / \langle E - 1 \rangle$ . Let  $\mathcal{A}_3$  be the  $\mathbb{C}[q, q^{-1}]$ -algebra (the Heisenberg-Weyl subalgebra of  $\mathcal{A}_4$  defined in 5.3.1) with generators  $\{c_+, c_-\}$  and the relation*

$$c_- \cdot c_+ - q^2 c_+ \cdot c_- = 1.$$

*$\mathcal{U}'_r(\mathfrak{h}_3)$  is  $\mathbb{C}$ -algebra isomorphic to  $\mathcal{A}_3$ .*

PROOF. The isomorphism is given by

$$A_- \mapsto qc_-, \quad A_+ \mapsto c_+, \quad r \mapsto q^{-1}.$$

□

**6.3.6.** There is an anti-automorphism  $\omega$  of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  given by

$$\begin{aligned} \omega(N) &= N, & \omega(A_+) &= A_-, & \omega(A_-) &= A_+, \\ \omega(E) &= E, & \omega(q) &= q, & \omega(r) &= r. \end{aligned}$$

The following maps  $\phi$  and  $\iota$  are automorphisms of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$

$$\begin{aligned} \phi(N) &= -N, & \phi(A_+) &= -A_+, & \phi(A_-) &= -A_-, \\ \phi(E) &= E, & \phi(q) &= q^{-1}, & \phi(r) &= r, \\ \iota(N) &= N, & \iota(A_+) &= -A_+, & \iota(A_-) &= A_-, \\ \iota(E) &= -E, & \iota(q) &= q, & \iota(r) &= r. \end{aligned}$$

**6.3.7.** Let  $m \in \mathbb{N}$ . The following identities hold in  $\mathcal{U}_{qr}(\mathfrak{h}_4)$

$$\begin{aligned} [N, A_+^m]_{q^m} &\equiv [m]_q A_+^m, \\ [A_-^m, N]_{q^m} &\equiv [m]_q A_-^m, \\ [A_-, A_+^m]_{r^m} &\equiv [m]_r E \cdot A_+^{m-1}. \end{aligned}$$

The proof is by induction on  $m$ .

#### 6.4. Fock module Representations

NOTATION. Let  $m \in \mathbb{N}$ . Define  $(m)_q := q^{-m} [m]_q \equiv \sum_{i=1}^m q^{1-2i}$ . Note that  $(0)_q = [0]_q = 0$ .

**6.4.1.** Let  $F$  be an infinite dimensional module over  $\mathbb{C}[q, q^{-1}, r, r^{-1}]$ , with basis  $\{u_n \mid n \in \mathbb{N}\}$

$$F := \sum_{n \in \mathbb{N}} \mathbb{C} [q, q^{-1}, r, r^{-1}] u_n.$$

Let  $c, j \in \mathbb{C}$ . Then the following action of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  on  $F$  makes  $F$  a  $\mathcal{U}_{qr}(\mathfrak{h}_4)$ -module

$$\begin{aligned} A_+ \cdot u_k &= u_{k+1}, \\ A_- \cdot u_k &= c(k)_r u_{k-1}, \\ N \cdot u_k &= (q^{-2k} j + (k)_q) u_k, \\ E \cdot u_k &= c u_k. \end{aligned}$$

Note that for the vacuum vector  $u_0$ :

$$\begin{aligned} A_- \cdot u_0 &= 0, \\ N \cdot u_0 &= j u_0, \\ \mathcal{U}_{qr}(\mathfrak{h}_4) \cdot u_0 &= F. \end{aligned}$$

Write  $F_{j,c}$  for  $F$ , when it is necessary to emphasise the dependence on  $j, c$ .



LEMMA 6.4.2. *If  $c \in \mathbb{C}^\times$  then the Fock module  $F$  is irreducible.*

PROOF. If  $c$  is nonzero, then for each  $v \in F$  there exists  $n \in \mathbb{N}$  such that  $A_-^n \cdot v = \alpha_v u_0$  ( $\alpha_v \in \mathbb{C}^\times$ ). Since  $u_0$  generates  $F$ , any other vector in  $F$ , can then be reached from the vacuum vector by applying a suitable polynomial in  $A_+$ .  $\square$

REMARK 6.4.3. Let  $j, j', c, c' \in \mathbb{C}$  be such that  $j' \neq j$  and/or  $c' \neq c$ . The Fock module  $F_{j,c}$  is inequivalent  $F_{j',c'}$ .

REMARK 6.4.4. A new representation, can be obtained from  $F$  by twisting by the automorphism  $\iota$ . It is inequivalent to  $F$  as a  $\mathcal{U}_{qr}(\mathfrak{h}_4)$ -module.

**6.4.5.** Since  $\mathcal{U}_r(\mathfrak{h}_3)$  is a subalgebra of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$ , many of the results on  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  that I mention in this chapter lead to corresponding results for the subalgebra  $\mathcal{U}_r(\mathfrak{h}_3)$ . For example the restriction to  $\mathcal{U}_r(\mathfrak{h}_3)$  of the  $\mathcal{U}_{qr}(\mathfrak{h}_4)$ -action on  $F$ , gives immediately a  $\mathcal{U}_r(\mathfrak{h}_3)$  Fock module. To avoid repetition I will generally only write down the corresponding results for  $\mathcal{U}_r(\mathfrak{h}_3)$  if they differ from those of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$ , otherwise they will be taken as understood.

**6.4.6.** In the quotient algebra  $\mathcal{U}_{qr}(\mathfrak{h}_4)/\langle q - r \rangle$  there exists a quadratic central element

$$A_+ \cdot A_- - E \cdot N.$$

It coincides in form with the quadratic central element of  $\mathfrak{h}_4$  (mentioned in 5.2.6).

## 6.5. The centre at roots of unity

**6.5.1.** Let  $\ell$  be a positive integer such that  $\ell > 2$ . Let  $\epsilon = \zeta$  be a primitive  $\ell$ -th root of unity. As usual define  $\ell'$  as

$$\ell' = \begin{cases} \ell & \text{if } \ell \text{ is odd,} \\ \frac{\ell}{2} & \text{if } \ell \text{ is even.} \end{cases}$$

LEMMA 6.5.2. *The following elements are central in  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$  ( $\mathcal{U}_\zeta(\mathfrak{h}_3)$ ) at the root of unity  $\epsilon = \zeta$*

$$\{A_+^{\ell'}, A_-^{\ell'}\}.$$

PROOF. The result follows directly from the relations in 6.3.7.  $\square$

$\mathcal{U}_\zeta(\mathfrak{h}_3)$  is finite dimensional over its centre, but  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$  is not since  $N$  does not generate a central element.

## 6.6. Fock module at roots of unity

**6.6.1.** Let  $F_\zeta$  denote the Fock module over  $\mathcal{U}_{q\zeta}(\mathfrak{h}_4)$  obtained by partial specialisation of the  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  Fock module  $F$  at  $r = \zeta$ .

LEMMA 6.6.2. *Consider the  $\mathcal{U}_{q\zeta}(\mathfrak{h}_4)$  Fock module  $F_\zeta$  at the root of unity  $\zeta$ . The vectors  $\{u_{k\ell'} \mid k \in \mathbb{Z}_{>0}\}$  are singular in  $F_\zeta$ . The module  $F_\zeta$  is reducible.*

**6.6.3.** Let  $k \in \mathbb{Z}_{>0}$ . Denote by  $F'_{k\ell'}$  the  $\mathcal{U}_{q\zeta}(\mathfrak{h}_4)$ -submodule generated by the singular vector  $u_{k\ell'}$ .

LEMMA 6.6.4. *The quotient module  $L_\zeta := F_\zeta/F'_{\ell'}$  is irreducible.*

LEMMA 6.6.5. *Let  $m, n \in \mathbb{N}$ , such that  $m < n$ .*

- (a) *The quotient module  $F'_{m\ell'}/F'_{n\ell'}$  is irreducible if and only if  $n = m + 1$ .*
- (b) *The quotient module  $F'_{m\ell'}/F'_{(m+1)\ell'}$  is equivalent to  $L_\zeta$ .*

PROOF. The proof is almost entirely the same as for 5.6.3.  $\square$

**6.6.6.** The irreducible quotient  $\mathcal{U}_{q\zeta}(\mathfrak{h}_4)$ -module  $L_\zeta$  is an example of a nilpotent representation of  $\mathcal{U}_{q\zeta}(\mathfrak{h}_4)$ :  $A_+$  and  $A_-$  act on it nilpotently.

**6.6.7.** In the specialisation  $\mathcal{U}_{1,r}(\mathfrak{h}_4)$  of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  at  $q = 1$ , the  $N$  has real eigenvalues and so it has an interpretation as a standard quantum mechanical number operator (rather than a  $q$ -number operator).

The algebra  $\mathcal{U}_{1,\zeta}$  and with its representation  $L_\zeta$  could be interpreted abstractly as a  $\zeta$ -parafermionic ( $\zeta$ -paragrassmann) oscillator.

### 6.7. Semicyclic and Cyclic representations

**6.7.1.** Let  $(\lambda, \mu) \in \mathbb{C} \times \{0\} \cup \mathbb{C} \times \{0\}$ . Let  $V_{\lambda,\mu} := \sum_{n=0}^{\ell'-1} \mathbb{C}v_n$  be a  $\mathbb{C}$ -vector space. At the root of unity  $\epsilon = \zeta$ , define the following  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$ -module structure on  $V_{\lambda,\mu}$

$$\begin{aligned} A_+ \cdot v_n &:= \begin{cases} v_{n+1} & \text{if } n \in [0, \ell' - 2] \\ \lambda v_0 & \text{if } n = \ell' - 1, \end{cases} \\ A_- \cdot v_n &:= \begin{cases} c(n)_\zeta v_{n-1} & \text{if } n \in [1, \ell' - 1] \\ \mu v_{\ell'-1} & \text{if } n = 0, \end{cases} \\ N \cdot v_n &:= (\epsilon^{-2k} j + (k)_\epsilon) v_n, \\ E \cdot v_n &= cv_n. \end{aligned}$$

Note that  $V_{0,0}$  ( $\lambda = 0$  and  $\mu = 0$ ) is equivalent to the quotient module  $L_{\epsilon\zeta}$ .

PROPOSITION 6.7.2. *The  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$ -module  $V_{\lambda,\mu}$  is irreducible.*

NOTATION. Let  $k \in \mathbb{Z}_{>0}$ . Define  $(k)_r! := \prod_{m \in [1,k]} (m)_r$  and  $(0)_r! := 1$ .

PROPOSITION 6.7.3 (SEMICYCLIC). *If  $(\lambda, \mu) \in \mathbb{C}^\times \times \{0\} \cup \{0\} \times \mathbb{C}^\times$  then  $V_{\lambda,\mu}$  is a semicyclic (semi-periodic)  $\mathcal{U}_\epsilon(\mathfrak{h}_4)$ -module.*

PROOF. The module is semicyclic, since  $A_+^{\ell'} \cdot v_n = \lambda v_n$ ,  $A_-^{\ell'} \cdot v_n = \mu c^{\ell'-1}(\ell' - 1)_\zeta! v_n$  ( $n \in [0, \ell' - 1]$ ): so only one of  $A_+$  and  $A_-$  acts cyclicly.  $\square$

REMARK 6.7.4. In the module  $V_{1,0}$  the restriction of the  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$ -action to the generator  $A_+$  gives a representation of the cyclic group  $Z_{\ell'}$ . Hence the action of  $A_+$  is called cyclic.

**6.7.5. Cyclic.** Let  $(\lambda, \mu) \in \mathbb{C} \times \mathbb{C}$ . Consider the following quotient of  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$  by a left ideal:

$$M_{\lambda, \mu, j} := \mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4) / \langle A_+^{\ell'} - \lambda, A_-^{\ell'} - \mu, N - j \rangle_L,$$

on which  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$  acts naturally on the left, giving a left  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$ -module.  $M_{\lambda, \mu, j}$  has the decomposition

$$M_{\lambda, \mu, j} = \sum_{n_{\pm} \in [0, \ell' - 1]} \mathbb{C} A_-^{m-} A_+^{n+}.$$

$M_{\lambda, \mu, j}$  is an irreducible  $\ell'^2$  dimensional cyclic  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$ -module. If  $(\lambda, \mu) \in (0, 0)$ , then  $M_{\lambda, \mu, j}$  is nilpotent. If  $(\lambda, \mu) \in (\mathbb{C}^\times \times \{0\}) \cup (\{0\} \times \mathbb{C}^\times)$ , then  $M_{\lambda, \mu, j}$  is semicyclic. If  $(\lambda, \mu) \in \mathbb{C}^\times \times \mathbb{C}^\times$ , then  $M_{\lambda, \mu, j}$  is cyclic.

## 6.8. Unitary representations

In this section I describe a (complex) unitary representation of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$ .

**6.8.1.** Define the sesquilinear scalar product  $(\cdot, \cdot)$  on the  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  Fock module  $F$ , to be contravariant with respect to the involutive  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  anti-automorphism  $\omega$  defined in 6.3.6 (with  $\omega(\alpha) := \alpha^*$  ( $\alpha \in \mathbb{C}$ ))

$$\begin{aligned} (u_k, u_l) &:= \delta_{k,l}(k)_r! & (k, l \in \mathbb{N}), \\ (x \cdot u_k, u_l) &= (u_k, \omega(x) \cdot u_l) & (x \in \mathcal{U}_{qr}(\mathfrak{h}_4)). \end{aligned}$$

LEMMA 6.8.2. *The triple  $(F, (\cdot, \cdot), \omega)$  is a  $\star$ -representation of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$ .*

PROPOSITION 6.8.3. *Let  $\epsilon \in \mathbb{R}^\times$  and  $\zeta \in \mathbb{R}_{>0}$ . Denote again by  $(\cdot, \cdot)$  the specialisation of the scalar product  $(\cdot, \cdot)$  on  $F$  to  $F_{\epsilon, \zeta}$ . The triple  $(F_{\epsilon, \zeta}, (\cdot, \cdot), \omega)$  is a unitary representation of  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$ .*

PROOF. If  $\zeta \in \mathbb{R}_{>0}$ , then  $[m]_\zeta$  is also real and positive. Therefore in this case the scalar product on  $F_{\epsilon\zeta}$  is positive definite. The condition on  $\epsilon$  ensures that  $(N \cdot v, N \cdot v) > 0$  ( $v \in F$ ).  $\square$

REMARK 6.8.4. It appears that there is no anti-automorphism of  $\mathcal{U}_{qr}(\mathfrak{h}_4)$  compatible with specialising  $q$  and  $r$  to a phase. So it is not possible to construct a unitary representation of  $\mathcal{U}_{\epsilon\zeta}(\mathfrak{h}_4)$  at a root of unity (c.f. 5.9.5).

## 6.9. Bosonisations of some quantum algebras with $\mathcal{U}_s(\mathfrak{h}_3)$

In this section I construct  $q$ -bosonisation homomorphisms of some well known finite and infinite dimensional quadratic quantum algebras into the algebra  $\mathcal{U}'_r(\mathfrak{h}_3)$ , defined by

$$\mathcal{U}'_r(\mathfrak{h}_3) := \mathcal{U}_r(\mathfrak{h}_3) / \langle E - 1 \rangle.$$

(In other words I construct realisations of these algebras in terms of the generators of  $\mathcal{U}'_r(\mathfrak{h}_3)$ .)

**6.9.1.** Let  $\alpha \in \mathbb{C}$ . The following map

$$\mathcal{U}'_{rr}(\mathfrak{h}_4) := \mathcal{U}_{qr}(\mathfrak{h}_4) / \langle q - r, E - 1 \rangle \rightarrow \mathcal{U}'_r(\mathfrak{h}_3)$$

is a  $\mathbb{C}[r, r^{-1}]$ -algebra homomorphism

$$\begin{aligned} A_+ &\mapsto A_+, \\ A_- &\mapsto A_-, \\ N &\mapsto \frac{1}{2}(A_+ \cdot A_- + A_- \cdot A_+ + \alpha). \end{aligned}$$

**6.9.2.** Let  $\mathcal{U}_{qr}(\mathfrak{sl}_2)$  denote the quantum algebra introduced in 6.2 and consider it here as a  $\mathbb{C}(q, r)$ -algebra. The following map

$$\mathcal{U}_{r^2, r^4}(\mathfrak{sl}_2) := \mathcal{U}_{q, r^4}(\mathfrak{sl}_2) / \langle q - r^2 \rangle \rightarrow \mathcal{U}'_r(\mathfrak{h}_3)$$

is a  $\mathbb{C}(r)$ -algebra homomorphism

$$\begin{aligned} W_0 &\mapsto ([2]_{r^2} [2]_r)^{-1} (r^2 A_- \cdot A_+ + r^{-2} A_+ \cdot A_-), \\ W_{\pm} &\mapsto \pm \left( ([2]_{r^2})^{\frac{1}{2}} [2]_r \right)^{-1} A_{\pm} \cdot A_{\pm}. \end{aligned}$$

**6.9.3.** The subset  $\{A_-, A_+^k \mid k \in \mathbb{N}\}$  of  $\mathcal{U}'_r(\mathfrak{h}_3)$  generates a subalgebra of  $\mathcal{U}'_r(\mathfrak{h}_3)$  with the relation  $[A_-, A_+^k]_{r^k} = [k]_r A_+^{k-1}$ . This subalgebra has a natural interpretation as the algebra of polynomials in one variable ( $A_+$ ) with a  $q$ -derivation ( $A_-$ ). From this point of view  $\mathcal{U}'_r(\mathfrak{h}_3)$  is the algebra of polynomials in a variable and a  $q$ -differential operator.

**6.9.4. Contractions.** Let  $\eta \in \mathbb{R}$ . Consider the following elements of the quantum algebra  $\mathcal{U}_{1, r^{-1}}(\mathfrak{sl}_2)$  (defined in 6.2)

$$\begin{aligned} X_0 &:= \eta^2 W_0, \\ X_{\pm} &:= \eta W_{\pm}. \end{aligned}$$

They satisfy the relations

$$\begin{aligned} [X_0, X_+] &= \eta^2 X_+, \\ [X_-, X_0] &= \eta^2 X_-, \\ [X_+, X_-]_r &= X_0. \end{aligned}$$

In the limit  $\eta \rightarrow 0$ , the subalgebra generated by  $\{X_0, X_+, X_-\}$  (a *contraction* of  $\mathcal{U}_{1, r^{-1}}(\mathfrak{sl}_2)$ ) is isomorphic to  $\mathcal{U}_r(\mathfrak{h}_3)$ . The isomorphism is given by

$$X_0 \mapsto E, \quad X_+ \mapsto A_-, \quad X_- \mapsto A_+.$$

A different contraction of  $\mathcal{U}_{q, r^{-1}}(\mathfrak{sl}_2)$  is also possible. Let  $\eta \in \mathbb{R}^{\times}$ . Consider the following elements in  $\mathcal{U}_{q, r^{-1}}(\mathfrak{sl}_2)$

$$\begin{aligned} Y_0 &:= W_0, \\ Y_{\pm} &:= \eta^{-1} W_{\pm}. \end{aligned}$$

They satisfy the relations

$$\begin{aligned} [Y_0, Y_+]_q &= Y_+, \\ [Y_-, Y_0]_q &= Y_-, \\ [Y_+, Y_-]_r &= \eta Y_0. \end{aligned}$$

In the limit as  $\eta \rightarrow 0$ , the contracted algebra generated by  $\{Y_0, Y_+, Y_-\}$  becomes a quadratic 2-parameter deformation of the Euclidean algebra.

**6.9.5.  $r$ -Witt algebra.** The  $r$ -Witt algebra  $\mathcal{U}_r(\mathcal{V}_0)$  (deformed centreless Virasoro algebra) is defined to be the  $\mathbb{C}[r, r^{-1}]$ -algebra with generators  $\{L_m \mid m \in \mathbb{Z}\}$  that satisfy the relations

$$[L_m, L_n]_{r^{n-m}} = [m - n]_r L_{m+n},$$

and certain complicated associativity relations (compare [Pol91]), which I will not write down here. (I thank Cosmas Zachos for bringing these to my attention.)  $\mathcal{U}_r(\mathcal{V}_0)$  is a deformation of the algebra of vector fields on the circle.

I define an extension  $\mathcal{U}_r(\mathfrak{h}_3^\vee)$  of  $\mathcal{U}'_r(\mathfrak{h}_3)$  by adjoining an (formal) generator  $A_+^{-1}$  with the relations

$$\begin{aligned} A_+^{-1} \cdot A_+ &= 1 = A_+ \cdot A_+^{-1}, \\ [A_+^{-1}, A_-]_r &= A_+^{-1} \cdot A_+^{-1}. \end{aligned}$$

The following map  $\mathcal{U}_r(\mathcal{V}_0) \rightarrow \mathcal{U}_r(\mathfrak{h}_3^\vee)$  is a  $\mathbb{C}[r, r^{-1}]$ -algebra homomorphism

$$L_m \mapsto -(A_+)^{1+m} \cdot A_- \quad (m \in \mathbb{Z}).$$

Note that the associativity relations are automatically satisfied, since  $\mathcal{U}_r(\mathfrak{h}_3^\vee)$  is associative.

**REMARK 6.9.6.** It is also possible to construct a centreless  $q$ -deformed  $W_\infty$  algebra [CKL90] (with generators  $W_n^{(k+1)} := A_+^{n+k} \cdot A_-^k$ ,  $k \in \mathbb{Z}_{>0}$ ). This was done in [GF91] using a  $q$ -Heisenberg algebra.

**6.9.7.** Note the following identity in  $\mathcal{U}_r(\mathfrak{h}_3^\vee)$

$$[A_+^{-m}, A_-]_{r^m} = [m]_r A_+^{-2m-1}.$$

Hence it follows that at the root of unity  $\zeta$ ,  $A_+^{-\ell'}$  lies in the centre of  $\mathcal{U}_\zeta(\mathfrak{h}_3^\vee)$ .

The following map  $\varphi : \mathcal{U}_\zeta(\mathcal{V}_0) \rightarrow \mathcal{U}_\zeta(\mathfrak{h}_3^\vee) / \langle A_+^{\ell'} - 1, A_+^{-\ell'} - 1, N \rangle_L$

$$\varphi(L_m) = -(A_+)^{1+m} \cdot A_- \quad (m \in \mathbb{Z}).$$

gives a finite representation of  $\mathcal{U}_\zeta(\mathcal{V}_0)$  with a cyclic property:  $\varphi(L_{m+k\ell'}) = \varphi(L_m)$  for all  $m, k \in \mathbb{Z}$ .

Similarly the map  $\vartheta : \mathcal{U}_\zeta(\mathcal{V}_0) \rightarrow \mathcal{U}_\zeta(\mathfrak{h}_3^\vee) / \langle A_+^{\pm\ell'} \rangle$  given by

$$\vartheta(L_m) := -(A_+)^{1+m} \cdot A_- \quad (m \in \mathbb{Z}).$$

gives a finite dimensional representation  $\vartheta(\mathcal{U}_\zeta(\mathcal{V}_0))$  with a nilpotent property.

**6.10. A Symmetry of  $\mathcal{U}_r(\mathfrak{h}_3)$** 

Let  $T := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a matrix, whose entries generate a quadratic  $\mathbb{C}[r, r^{-1}]$ -bialgebra  $\mathfrak{A}$ , with relations that will be given presently and coproduct  $\Delta(T) := T \dot{\otimes} T$ .

**PROPOSITION 6.10.1.** *Consider the left coaction  $\Delta_L : \mathcal{U}_r(\mathfrak{h}_3) \rightarrow \mathfrak{A} \otimes \mathcal{U}_r(\mathfrak{h}_3)$  of  $\mathfrak{A}$  on  $\mathcal{U}_r(\mathfrak{h}_3)$ :*

$$\begin{pmatrix} A_+ \\ A_- \end{pmatrix} \mapsto T \dot{\otimes} \begin{pmatrix} A_+ \\ A_- \end{pmatrix}, \quad E \mapsto E' := 1 \otimes E.$$

*If the elements of  $T$  satisfy the following relations, then the left coaction  $\Delta_L$  is a  $\mathbb{C}[r, r^{-1}]$ -algebra homomorphism*

$$\begin{aligned} ac &= r^2 ca, & ad - r^2 cb &= 1, \\ bd &= r^2 db, & da - r^{-2} bc &= 1. \end{aligned}$$

**PROPOSITION 6.10.2.** *The right coaction  $\Delta_R : \mathcal{U}_r(\mathfrak{h}_3) \rightarrow \mathcal{U}_r(\mathfrak{h}_3) \otimes \mathfrak{A}$  of  $T$  on  $\mathcal{U}_r(\mathfrak{h}_3)$ , given by:*

$$\begin{pmatrix} A_+ & A_- \end{pmatrix} \mapsto \begin{pmatrix} A_+ & A_- \end{pmatrix} \dot{\otimes} T, \quad E \mapsto E \otimes 1$$

*is an algebra homomorphism, if the elements of  $T$  satisfy the relations*

$$\begin{aligned} ab &= r^2 ba, & ad - r^2 bc &= 1, \\ cd &= r^2 dc, & da - r^{-2} cb &= 1. \end{aligned}$$

**PROPOSITION 6.10.3.** *If  $\mathcal{U}_r(\mathfrak{h}_3)$  is covariant with respect to both the left and right  $\mathfrak{A}$ -coactions  $\Delta_L$  and  $\Delta_R$ , then the generators of  $\mathfrak{A}$  satisfy the relations of  $\text{fun}_{r^2}(SL_2)$ .*

**PROOF.** Combining the relations from the two previous propositions forces the additional relation  $bc = cb$  in  $\mathfrak{A}$ . Then all the relations given coincide with those of  $\text{fun}_{r^2}(SL_2)$ .  $\square$

**6.10.4.** By construction the maps  $\Delta_L$  and  $\Delta_R$  are compatible with the coproduct of  $\mathfrak{A}$ :

$$\begin{aligned} (\Delta_L \otimes \text{id}) \circ \Delta_L &= (\text{id} \otimes \Delta) \circ \Delta_L, \\ (\text{id} \otimes \Delta_R) \circ \Delta_R &= (\Delta \otimes \text{id}) \circ \Delta_R. \end{aligned}$$

and the two coactions cocommute

$$(\text{id} \otimes \Delta_R) \circ \Delta_L = (\Delta_L \otimes \text{id}) \circ \Delta_R.$$

## CHAPTER 7

# Quantum affine algebras

### 7.1. Introduction

To every affine Cartan matrix of an affine (Kac-Moody) Lie algebra  $\hat{\mathfrak{g}}$ , Drinfeld and Jimbo have associated a Hopf algebra  $\mathcal{U}_q(\hat{\mathfrak{g}})$ , the quantum affine (Kac-Moody) algebra or quantum group of  $\hat{\mathfrak{g}}$ . In this chapter I describe the quantum affine algebra  $\mathcal{U}_q(\hat{\mathfrak{g}})$ , when  $\hat{\mathfrak{g}}$  is an untwisted affine Kac-Moody algebra.

In the classical theory of Kac-Moody algebras and their applications, a very important role is played by their centrally extended loop algebra presentation. It has a number of advantages over the Chevalley presentation. In particular the loop algebra presentation  $\hat{\mathfrak{g}}$  makes it manifest that the algebra is infinite dimensional and that it has a central element. In the case of quantum affine Kac-Moody algebras such a loop algebra presentation was given by Drinfeld [Dri88]. This presentation has been extensively utilised in applications of quantum affine algebras and in their representation theory (see for example [DFJMN93, CP91b]). Unfortunately Drinfeld did not give a proof that his loop algebra presentation is isomorphic to the usual Chevalley presentation. Another related problem is the construction of a basis of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  in terms of the underlying root system, like the one described for  $\mathcal{U}_q(\mathfrak{g})$  in 4.6. This problem is much harder in the affine case because the root system and affine Weyl group  $\hat{W}$  of  $\hat{\mathfrak{g}}$  are both infinite dimensional: in particular there is no unique longest element of  $\hat{W}$  that gives a natural basis of root vectors in  $\mathcal{U}_q(\hat{\mathfrak{g}})$  as there is in the finite case and also because of the existence of imaginary root vectors.

The isomorphism between the two presentations was proved recently by Beck [Bec93], and Khoroshkin and Tolstoy [KT93b] and their work [KT93a, Bec94] also led to Poincaré-Birkhoff-Witt type bases of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ . Beck's work in particular gave an extension of Lusztig's braid group action [Lus90c, Lus93] on  $\mathcal{U}_q(\hat{\mathfrak{g}})$  to the group of automorphisms  $\hat{B}$  associated to the extended affine Weyl group of  $\hat{\mathfrak{g}}$ . The action of  $\hat{B}$  on the Chevalley generators, corresponding to the underlying  $\mathcal{U}_q(\mathfrak{g})$  subalgebra, generates all the real root vectors of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ . There is a subgroup  $\mathcal{P}$  of  $\hat{B}$  corresponding to the lattice of translations in the extended affine Weyl group of  $\hat{\mathfrak{g}}$ . The imaginary root vectors on the other hand cannot be reached by the action  $\hat{B}$ . The imaginary root vectors are fixed under the action of  $\mathcal{P}$ .

As in the classical case,  $\mathcal{U}_q(\hat{\mathfrak{g}})$  has a Heisenberg subalgebra  $\mathcal{U}_q(\mathfrak{h})$  generated by its

imaginary root vectors.  $\mathcal{U}_q(\mathfrak{H})$  has a similar representation theory to the classical Heisenberg subalgebra. It has highest weight Fock modules which are irreducible. At a real positive specialisation the Fock modules are unitarisable.

## 7.2. Preliminaries

I start by recalling the definition of a quantum affine Kac-Moody algebra due to Jimbo [Jim85] and Drinfeld [Dri87].

**7.2.1.** Let  $\mathfrak{g}$  be a simple Lie algebra of rank  $r$  with Cartan matrix  $(a_{ij})_{i,j \in [1,r]}$ . Let  $(a_{ij})_{i,j \in [0,r]}$  be the extended affine Cartan matrix of the (untwisted) Kac-Moody algebra  $\hat{\mathfrak{g}}$  of  $\mathfrak{g}$ , in the sense of Kac [Kac90], so that  $a_{ii} = 2$  and  $a_{ij} \leq 0$  for  $i \neq j$ . Fix  $r+1$  positive coprime integers  $(d_i)_{i \in [0,r]}$ , such that  $(d_i a_{ij})$  is a symmetric matrix. The Cartan matrix  $(a_{ij})_{i,j \in [0,r]}$  of  $\hat{\mathfrak{g}}$  has rank  $r$ :  $\det((a_{ij})_{i,j \in [0,r]}) = 0$ .

**DEFINITION 7.2.2.** The quantum affine (Kac-Moody) algebra  $\mathcal{U}_q(\hat{\mathfrak{g}})$  is the associative  $\mathbb{C}(q)$ -algebra with  $\mathbf{1}$  and generators  $\{e_i, f_i, k_i^{\pm 1}, D \mid i \in [0, r]\}$  with the relations:

$$\begin{aligned} k_i \cdot k_i^{-1} &= 1 = k_i^{-1} \cdot k_i, & k_i \cdot k_j &= k_j \cdot k_i, \\ k_i \cdot e_j \cdot k_i^{-1} &= q_i^{a_{ij}} e_j, & k_i \cdot f_j \cdot k_i^{-1} &= q_i^{-a_{ij}} f_j, \\ e_i \cdot f_j - f_j \cdot e_i &= \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, & D \cdot k_i &= k_i \cdot D, \\ D \cdot e_i \cdot D^{-1} &= q^{\delta_{i0}} e_i, & D \cdot f_i \cdot D^{-1} &= q^{-\delta_{i0}} f_i, \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} e_i^n \cdot e_j \cdot e_i^{1-a_{ij}-n} &= 0 \quad (i \neq j), \\ \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} f_i^n \cdot f_j \cdot f_i^{1-a_{ij}-n} &= 0 \quad (i \neq j). \end{aligned}$$

The generators  $\{e_i, f_i \mid i \in [0, r]\}$  are called the Chevalley generators.

**7.2.3.** The quantum enveloping algebra  $\mathcal{U}_q(\hat{\mathfrak{g}})$  is a Hopf algebra. The Hopf algebra structure is as follows.

The coproduct map  $\Delta : \mathcal{U}_q(\hat{\mathfrak{g}}) \rightarrow \mathcal{U}_q(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q(\hat{\mathfrak{g}})$  is

$$\begin{aligned} \Delta : k_i &\mapsto k_i \otimes k_i, \\ \Delta : e_i &\mapsto e_i \otimes \mathbf{1} + k_i \otimes e_i, \\ \Delta : f_i &\mapsto f_i \otimes k_i^{-1} + \mathbf{1} \otimes f_i, \\ \Delta : D &\mapsto D \otimes D. \end{aligned}$$

The antipode map  $S : \mathcal{U}_q(\hat{\mathfrak{g}}) \rightarrow \mathcal{U}_q(\hat{\mathfrak{g}})$  is

$$S : k_i \mapsto k_i^{-1}, \quad S : e_i \mapsto -k_i^{-1} e_i, \quad S : f_i \mapsto -f_i k_i, \quad S : D \mapsto D^{-1}.$$

The counit map  $\epsilon : \mathcal{U}_q(\hat{\mathfrak{g}}) \rightarrow \mathbb{C}(q)$  is

$$\epsilon : k_i \mapsto 1, \quad \epsilon : e_i \mapsto 0, \quad \epsilon : f_i \mapsto 0, \quad \epsilon : D \mapsto 1.$$



**7.2.4. Triangular decomposition.** The Cartan subalgebra  $\mathcal{U}_q(\hat{\mathfrak{h}})$  is generated by  $\{k_i, D\}_{i \in [0, r]}$ . The (positive) Chevalley generators  $\{e_i\}_{i \in [0, r]}$  generate the positive roots subalgebra  $\mathcal{U}_q(\hat{\mathfrak{n}}_+)$  and the (negative) Chevalley generators  $\{f_i\}_{i \in [0, r]}$  generate the negative roots subalgebra  $\mathcal{U}_q(\hat{\mathfrak{n}}_-)$ . As in the finite case [Ros88],  $\mathcal{U}_q(\hat{\mathfrak{g}})$  has a triangular decomposition  $\mathcal{U}_q(\hat{\mathfrak{g}}) \simeq \mathcal{U}_q(\hat{\mathfrak{n}}_-) \otimes \mathcal{U}_q(\hat{\mathfrak{h}}) \otimes \mathcal{U}_q(\hat{\mathfrak{n}}_+)$  (vector space isomorphism by multiplication).

**7.2.5. Derived algebra.** The derived quantum affine algebra  $\mathcal{U}_q(\hat{\mathfrak{g}}')$ , which is the quantum enveloping algebra of the derived affine algebra  $\hat{\mathfrak{g}}'$ , is the subalgebra of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  generated by  $\{e_i, k_i^{\pm 1}, f_i\}_{i \in [0, r]}$  (i.e. excluding  $D^{\pm 1}$ ). Let  $\mathcal{U}_q(\hat{\mathfrak{n}}'_+)$ ,  $\mathcal{U}_q(\hat{\mathfrak{h}}')$  and  $\mathcal{U}_q(\hat{\mathfrak{n}}'_-)$  be the positive roots subalgebra, Cartan subalgebra and negative roots subalgebra of  $\mathcal{U}_q(\hat{\mathfrak{g}}')$ , generated respectively by  $\{e_i\}$ ,  $\{k_i\}$  and  $\{f_i\}$ . There is a corresponding triangular decomposition  $\mathcal{U}_q(\hat{\mathfrak{g}}') = \mathcal{U}_q(\hat{\mathfrak{n}}'_-) \otimes \mathcal{U}_q(\hat{\mathfrak{h}}') \otimes \mathcal{U}_q(\hat{\mathfrak{n}}'_+)$ . Clearly  $\mathcal{U}_q(\hat{\mathfrak{n}}'_+) \simeq \mathcal{U}_q(\hat{\mathfrak{n}}_+)$  and  $\mathcal{U}_q(\hat{\mathfrak{n}}'_-) \simeq \mathcal{U}_q(\hat{\mathfrak{n}}_-)$ , but  $\mathcal{U}_q(\hat{\mathfrak{h}})$  is an extension of  $\mathcal{U}_q(\hat{\mathfrak{h}}')$  by the grading generator  $D$ .  $D$  is called the *derivation* or scaling element of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ .

NOTATION. For  $\epsilon \in \mathbb{C}^\times$ , define  $\epsilon_i := \epsilon^{d_i}$ . Denote by  $[n]_\epsilon$  and  $\begin{bmatrix} n \\ m \end{bmatrix}_\epsilon$  in  $\mathbb{C}$  the specialisation of the  $q$ -numbers  $[n]_q$  and  $\begin{bmatrix} n \\ m \end{bmatrix}_q$  in  $\mathbb{C}[q, q^{-1}]$  at  $q = \epsilon$ .

DEFINITION 7.2.6 (SPECIALISATION). Define the following elements in  $\mathcal{U}_q(\hat{\mathfrak{g}})$

$$h_i := \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \quad \text{and} \quad d := \frac{D - D^{-1}}{q - q^{-1}}.$$

Define  $\mathcal{U}_A(\hat{\mathfrak{g}})$  to be the  $\mathbb{C}[q, q^{-1}]$ -subalgebra of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  generated by

$$\{e_i, f_i, k_i^{\pm 1}, D^{\pm 1}, h_i, d\}.$$

For  $\epsilon \in \mathbb{C}^\times$ , I define the specialisation of  $\mathcal{U}_A(\hat{\mathfrak{g}})$  at  $q = \epsilon$  to be  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}}) := \mathcal{U}_A(\hat{\mathfrak{g}})/\langle q - \epsilon \rangle$ .

**7.2.7.** The affine Lie algebra  $\hat{\mathfrak{g}}$  associated to the affine Cartan matrix  $(a_{ij})_{i, j \in [0, r]}$  has generators  $\{\bar{h}_i, \bar{e}_i, \bar{f}_i, \bar{d} \mid i \in [0, r]\}$ , which satisfy the following relations (Chevalley presentation)

$$\begin{aligned} [\bar{h}_i, \bar{h}_j] &= 0, & [\bar{e}_i, \bar{f}_j] &= \delta_{ij} \bar{h}_i, \\ [\bar{h}_i, \bar{e}_j] &= a_{ij} \bar{e}_j, & [\bar{h}_i, \bar{f}_j] &= -a_{ij} \bar{f}_j, \\ [\bar{d}, \bar{h}_i] &= 0, & [\bar{d}, \bar{e}_i] &= \delta_{i0} \bar{e}_i, \\ [\bar{d}, \bar{f}_i] &= -\delta_{i0} \bar{f}_i, \\ \text{ad}(\bar{e}_i)^{1-a_{ij}}(\bar{e}_j) &= 0, \quad \text{ad}(\bar{f}_i)^{1-a_{ij}}(\bar{f}_j) = 0. \end{aligned}$$

Let  $\mathcal{U}(\hat{\mathfrak{g}})$  be the universal enveloping algebra  $\hat{\mathfrak{g}}$ .

PROPOSITION 7.2.8. [Lus88, DCK90] Let  $\mathcal{U}_1(\hat{\mathfrak{g}})$  be the specialisation of  $\mathcal{U}_A(\hat{\mathfrak{g}})$  at  $q = 1$ . The following map  $\mathcal{U}_1(\hat{\mathfrak{g}})/\langle k_i - 1, D - 1 \mid i \in [0, r] \rangle \longrightarrow \mathcal{U}(\hat{\mathfrak{g}})$  is a  $\mathbb{C}$ -algebra isomorphism

$$\begin{aligned} e_i &\mapsto \bar{e}_i, & f_i &\mapsto \bar{f}_i, \\ h_i &\mapsto \bar{h}_i, & d &\mapsto \bar{d}. \end{aligned}$$

REMARK 7.2.9. Fix  $i \in [0, r]$ . The elements  $\{e_i, k_i^{\pm 1}, f_i\}$  in  $\mathcal{U}_q(\hat{\mathfrak{g}})$  generate a  $\mathcal{U}_q(\mathfrak{sl}_2)$  subalgebra of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ .

The generators  $\{e_i, (k_i)^{\pm 1}, f_i \mid i \in [1, r]\}$  generate a  $\mathcal{U}_q(\mathfrak{g})$  subalgebra of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ .

**7.2.10.** Let  $(a_i)_{i \in [0, r]} \in \mathbb{Z}_{>0}^{r+1}$  be the unique vector of coprime positive integers that satisfy  $\sum_{j=0}^r a_{ij} a_j = 0$  (with  $a_0 = 1$ ). Then  $\{a_0, \dots, a_r\}$  are the Coxeter labels (or Kac numbers) of the Dynkin diagram of  $\hat{\mathfrak{g}}$ . Recall that the canonical central element of  $\hat{\mathfrak{g}}$  is given by  $\sum_{i=0}^r a_i^\vee \bar{h}_i$ , where  $\{a_0^\vee, \dots, a_r^\vee\}$  are the dual Coxeter labels (dual Kac numbers) of  $\hat{\mathfrak{g}}$  satisfying  $\sum_{i=0}^r a_i^\vee a_{ij} = 0$  ( $a_i^\vee = d_i a_i$ ).

LEMMA 7.2.11. The canonical central element of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  is

$$C := \prod_{i \in [0, r]} k_i^{a_i}$$

PROOF. It is easily checked that  $C$  commutes with all the Chevalley generators of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ . I show that  $C$  commutes with  $e_j$ . A similar simple calculation shows that  $C$  and  $f_j$  commute.

$$\begin{aligned} C \cdot e_i &= \prod_{i=0}^r k_i^{a_i} \cdot e_j \\ &= q^{\sum_{i=0}^r a_i d_i a_{ij}} e_j \cdot \prod_{i=0}^r k_i^{a_i} \\ &= e_j \cdot C. \end{aligned}$$

The last step follows from the symmetry of the symmetrised Cartan matrix:  $d_i a_{ij} = d_j a_{ji}$  and the definition of the Coxeter labels.  $\square$

**7.2.12. Gradations.** Let  $(M, +)$  be an abelian group. Then an  $M$ -gradation of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  is a decomposition with respect to  $M$ :

$$\mathcal{U}_q(\hat{\mathfrak{g}}) := \bigoplus_{\alpha \in M} \mathcal{U}_q(\hat{\mathfrak{g}})_{(\alpha)},$$

such that  $\forall \alpha, \beta \in M$ ,  $\mathcal{U}_q(\hat{\mathfrak{g}})_{(\alpha)} \cdot \mathcal{U}_q(\hat{\mathfrak{g}})_{(\beta)} \subseteq \mathcal{U}_q(\hat{\mathfrak{g}})_{(\alpha+\beta)}$ . An element  $x \in \mathcal{U}_q(\hat{\mathfrak{g}})_{(\alpha)}$  is said to have degree  $\alpha$  ( $\deg x = \alpha$ ).

Let  $s := (s_0, \dots, s_r) \in \mathbb{Z}^{r+1}$ . Then

$$\begin{aligned} \deg e_i &:= s_i, & \deg f_i &:= -s_i, \\ \deg k_i &:= 0, & \deg D &:= 0, \end{aligned}$$

defines a  $(\mathbb{Z}, +)$ -gradation of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ , (which in the classical case, Kac calls a  $\mathbb{Z}$ -gradation of type  $s$ ). For example choosing  $s_i = 1$  ( $i \in [0, r]$ ) gives the principal gradation of

$\mathcal{U}_q(\hat{\mathfrak{g}})$ . Choosing instead  $s_0 = 1$  and  $s_i = 0$  ( $i \in [1, r]$ ) gives the homogeneous gradation of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ .

Let  $\hat{Q} := \sum_{i \in [0, r]} \mathbb{Z}\alpha_i$  be the root lattice of  $\hat{\mathfrak{g}}$  (see 7.3.2). There is a  $\hat{Q}$ -gradation of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  given by

$$\begin{aligned} \deg e_i &:= \alpha_i, & \deg f_i &:= -\alpha_i, \\ \deg k_i &:= 0, & \deg D &:= 0. \end{aligned}$$

LEMMA 7.2.13. *The map  $\omega : \mathcal{U}_q(\hat{\mathfrak{g}}) \rightarrow \mathcal{U}_q(\hat{\mathfrak{g}})$*

$$\begin{aligned} k_i &\mapsto k_i^{-1}, & D &\mapsto D^{-1}, & q &\mapsto q^{-1}, \\ e_i &\mapsto f_i, & f_i &\mapsto e_i, \end{aligned}$$

*extends as an  $\mathbb{C}$ -algebra anti-automorphism to the whole of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ . It is called the Cartan involution of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ .*

*The following map  $\varphi : \mathcal{U}_q(\hat{\mathfrak{g}}) \rightarrow \mathcal{U}_q(\hat{\mathfrak{g}})$  is a  $\mathbb{C}$ -algebra automorphism of  $\mathcal{U}_q(\hat{\mathfrak{g}})$*

$$\begin{aligned} k_i &\mapsto k_i, & D &\mapsto D, & q &\mapsto q^{-1}, \\ e_i &\mapsto f_i, & f_i &\mapsto e_i. \end{aligned}$$

### 7.3. Lattices and Weyl group

**7.3.1.** Recall from 4.3 the definitions of the following structures associated to  $\mathfrak{g}$ . The weight lattice  $P$  of  $\mathfrak{g}$  over  $\mathbb{Z}$  has generators  $\omega_i$  ( $i \in [1, r]$ )

$$P := \sum_{i \in [1, r]} \mathbb{Z}\omega_i.$$

The simple roots of  $\mathfrak{g}$  are  $\alpha_i := \sum_{j \in [1, r]} a_{ij}\omega_j$  ( $i \in [1, r]$ ) and the root lattice of  $\mathfrak{g}$  is the sub-lattice of  $P$  generated over  $\mathbb{Z}$  by  $\{\alpha_i \mid i \in [1, r]\}$

$$Q := \sum_{i \in [1, r]} \mathbb{Z}\alpha_i.$$

The coroot lattice is the lattice dual to  $P$  over  $\mathbb{Z}$ :  $Q^\vee := \text{Hom}(P, \mathbb{Z})$ , with a basis  $\alpha_i^\vee$  ( $i \in [1, r]$ ) with the pairing  $\langle \cdot, \cdot \rangle : P \times Q^\vee \rightarrow \mathbb{Z}$  given by

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}.$$

Then  $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$ .

Let  $W$  be the Weyl group associated to  $\mathfrak{g}$  with generators  $s_i$  (the fundamental reflections) as defined in 4.3.6. The Weyl group  $W$  acts on  $P$  as

$$s_i : x \mapsto x - \langle x, \alpha_i^\vee \rangle \alpha_i \quad (x \in P).$$

$Q$  is invariant under the action of  $W$ . The set of simple roots is  $\Pi := \{\alpha_i \mid i \in [1, r]\}$ . The root system  $R$  of  $\mathfrak{g}$  is the subset of  $Q$  given by the  $W$ -orbit of  $\Pi$ ,  $R := W \cdot \Pi$ .  $P_+ := \sum_{i \in [1, r]} \mathbb{N}\omega_i$  and  $Q_+ := \sum_{i \in [1, r]} \mathbb{N}\alpha_i$ . The set of positive roots of  $\mathfrak{g}$  is  $R_+ := R \cap Q_+$ .

The coroot lattice  $Q^\vee$  can be embedded in the coweight lattice  $P^\vee$ , which is the dual lattice to  $Q$ ,  $P^\vee := \text{Hom}(Q, \mathbb{Z})$ , with basis  $\{\omega_i^\vee \mid i \in [1, r]\}$  and pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle' : Q \times P^\vee &\rightarrow \mathbb{Z}, \\ \langle \alpha_i, \omega_j^\vee \rangle' &= \delta_{ij}. \end{aligned}$$

The embedding is given by  $\alpha_i^\vee = \sum_{j \in [1, r]} a_{ij} \omega_j^\vee$ . Note that the two pairings coincide on  $Q \times Q^\vee$ :

$$\langle \cdot, \cdot \rangle_{|Q \times Q^\vee} = \langle \cdot, \cdot \rangle'_{|Q \times Q^\vee}.$$

Let  $\Pi^\vee := \{\alpha_i^\vee \mid i \in [1, r]\}$  be the set of simple coroots and  $R^\vee := W \cdot \Pi^\vee$  be the coroot system of  $\mathfrak{g}$ . The correspondence  $\alpha_i \leftrightarrow \alpha_i^\vee$  ( $i \in [1, r]$ ) extends to  $R \leftrightarrow R^\vee$ . The Weyl group acts on  $P^\vee$  by

$$s_i : x^\vee \mapsto x^\vee - \langle \alpha_i, x^\vee \rangle' \alpha_i^\vee \quad (x^\vee \in P^\vee).$$

The coroot lattice  $Q^\vee$  is invariant under the action of  $W$ :

$$s_i : \alpha_j^\vee \mapsto \alpha_j^\vee - \langle \alpha_i, \alpha_j^\vee \rangle \alpha_j^\vee.$$

From 4.3.11 it follows that  $\langle \alpha, \alpha^\vee \rangle = 2$  for all  $\alpha \in R$ .

**7.3.2. Affine Weyl group.** The affine Weyl group of  $\hat{\mathfrak{g}}$  associated to  $(a_{ij})_{i,j \in [0, r]}$  has generators  $\{s_i \mid i \in [0, r]\}$  and the relations of 4.3.6 (with  $m_{ij} = \infty$  if  $\frac{a_{ij}a_{ji}}{4} \geq 1$ ).

Let  $\hat{P} := \sum_{i \in [0, r]} \mathbb{Z} \omega_i$  be the affine weight lattice of  $\hat{\mathfrak{g}}$  with basis  $\{\omega_i \mid i \in [0, r]\}$ . Let  $\{\alpha_i := \sum_{j \in [0, r]} a_{ij} \omega_j \mid i \in [0, r]\}$  be the simple roots of  $\hat{\mathfrak{g}}$  ( $\alpha_0$  is the extra root). Define the affine root lattice of  $\hat{\mathfrak{g}}$  to be  $\hat{Q} := \sum_{i \in [0, r]} \mathbb{Z} \alpha_i$ .

**7.3.3. Symmetric form.** I define a bilinear map  $(\cdot, \cdot) : \hat{P} \times \hat{Q} \rightarrow \mathbb{Z}$

$$(\omega_i, \alpha_j) := d_i \delta_{ij}.$$

Note that the restriction of this pairing to  $\hat{Q} \times \hat{Q}$  defines a symmetric  $\mathbb{Z}$ -valued bilinear form on  $\hat{Q}$ , since

$$(\alpha_i, \alpha_j) = d_i a_{ij}.$$

**7.3.4. Extended affine Weyl group.** Following [Bec93], I make the following definitions.

Define the extended affine Weyl group  $\hat{W} := W \ltimes P^\vee$ , where the semi-direct product is defined using the action of  $W$  on  $P^\vee$  to be

$$(w_1, x_1) \cdot (w_2, x_2) := (w_1 w_2, w_2^{-1}(x_1) + x_2) \quad (\forall w_1, w_2 \in W; x_1, x_2 \in P^\vee).$$

The identity element in  $\hat{W}$  is  $(1, 0)$  and for each  $(w, x) \in \hat{W}$  the inverse is  $(w^{-1}, -w^{-1}(x))$  ( $w \in W, x \in P^\vee$ ). Write  $w$  for  $(w, 0) \in \hat{W}$  ( $w \in W$ ). Also write  $x$  for  $(1, x)$  ( $x \in P^\vee$ ).

**NOTATION.** Let  $\alpha \in R$ . Define  $s_\alpha$  to act on  $x \in P$  as  $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$  and to act on  $x^\vee \in P^\vee$  as  $s_\alpha(x^\vee) = x^\vee - \langle \alpha, x^\vee \rangle' \alpha^\vee$ .

Denote by  $\theta := \sum_{i \in [1, r]} a_i \alpha_i$  the highest root in  $R_+$  of  $\mathfrak{g}$  and  $\theta^\vee$  the highest coroot of  $\mathfrak{g}$ . Let  $s_0 = (s_\theta, \theta^\vee) \in \hat{W}$ . The set  $\{s_i \mid i \in [0, r]\}$  of  $\hat{W}$  generates the affine Weyl group  $\tilde{W}$ . Therefore  $\tilde{W}$  is a subgroup of  $\hat{W}$ : in fact  $\tilde{W}$  is a normal (Coxeter) subgroup of  $\hat{W}$ . The quotient group  $\mathcal{T} := \hat{W}/\tilde{W}$  is a finite group (a certain subgroup of Dynkin diagram automorphisms of  $\hat{\mathfrak{g}}$ ) [Bou68, VI]. A diagram automorphism  $\tau \in \mathcal{T}$  acts on  $\tilde{W}$  as  $\tau s_i \tau^{-1} = s_{\tau(i)}$ . Then I can identify  $\hat{W} = \mathcal{T} \ltimes \tilde{W}$ . In  $\mathcal{T} \ltimes \tilde{W}$  the product is given by  $(\tau_1, w_1) \cdot (\tau_2, w_2) := (\tau_1 \tau_2, \tau_2^{-1}(w_1)w_2)$ .

Define the canonical imaginary root of  $\hat{\mathfrak{g}}$  to be

$$\delta := \alpha_0 + \theta = \sum_{i \in [0, r]} a_i \alpha_i.$$

The pairing  $\langle \cdot, \cdot \rangle : P \times Q^\vee \rightarrow \mathbb{Z}$  extends naturally to a pairing  $\langle \cdot, \cdot \rangle : \hat{P} \times \hat{Q}^\vee \rightarrow \mathbb{Z}$  and likewise the pairing  $\langle \cdot, \cdot \rangle' : Q \times P^\vee$  to  $\hat{Q} \times \hat{P}^\vee \rightarrow \mathbb{Z}$ .

The extended affine Weyl group  $\hat{W}$  acts on  $\hat{Q}$ : in particular the elements of  $P^\vee$  act on  $\hat{Q}$  as

$$x(\beta) = \beta - \langle \beta, x \rangle' \delta \quad (\beta \in Q, x \in P^\vee).$$

**7.3.5. Root system.** Let  $R$  be the (finite) root system of  $\mathfrak{g}$ . Recall that the root system  $\hat{R}$  of  $\hat{\mathfrak{g}}$  is infinite dimensional and has the following structure.

There are real roots and imaginary roots. The set of real roots is

$$\hat{R}^{\text{re}} := \{\alpha + n\delta \mid \alpha \in R, n \in \mathbb{Z}\}.$$

The set of imaginary roots is

$$\hat{R}^{\text{im}} := \{n\delta \mid n \in \mathbb{Z}^\times\}.$$

The real roots are generated by the action of  $\hat{W}$  on the simple roots,  $\hat{R}^{\text{re}} = \hat{W} \cdot \Pi$ . The imaginary roots are fixed by  $\tilde{W}$ :  $s_i(n\delta) = n\delta$  (for all  $i \in [0, r]$  and  $n \in \mathbb{Z}$ ), since  $\langle \delta, \alpha_i^\vee \rangle = 0$  ( $i \in [0, r]$ ). The positive real roots are

$$\hat{R}_+^{\text{re}} := \hat{R}^{\text{re}} \cap \hat{Q}_+ = \{\beta + n\delta \mid \beta \in R, n \in \mathbb{Z}_{>0}\} \cup \{\alpha \mid \alpha \in R_+\}.$$

The positive imaginary roots are

$$\hat{R}_+^{\text{im}} = \hat{R}^{\text{im}} \cap Q_+ = \{n\delta \mid n \in \mathbb{Z}_{>0}\}.$$

Finally the root system and positive roots are

$$\begin{aligned} \hat{R} &:= \hat{R}^{\text{re}} \cup \hat{R}^{\text{im}}, \\ \hat{R}_+ &:= \hat{R} \cap Q_+ = \hat{R}_+^{\text{re}} \cup \hat{R}_+^{\text{im}}. \end{aligned}$$

**7.3.6. Ordering of  $\hat{R}_+$ .** Recall 4.5.5 that a reduced expression of the longest element  $w_0 \in W$  gives an ordering in  $R_+$  and a corresponding ordering of  $-R_+$ . Defining also  $\alpha + m\delta > \alpha + n\delta$ , if  $m > n$  ( $\alpha \in R \cup \{0\}$ ,  $m, n \in \mathbb{Z}_{>0}$ ) and  $\beta + n\delta > m\delta > -\beta + p\delta$  ( $\beta \in R_+$ ,  $m, n, p \in \mathbb{Z}_{>0}$ ), gives then a ordering of  $\hat{R}_+$ .

### 7.4. The braid group action

The results from this section will be used in 8.3 when a quantum affine algebra at a root of unity is considered.

NOTATION. I define for each  $n \in \mathbb{N}$  and  $i \in [0, r]$  the following elements in  $\mathcal{U}_q(\hat{\mathfrak{g}})$

$$e_i^{(n)} := \frac{e_i^n}{[n]_{q_i}!} \quad \text{and} \quad f_i^{(n)} := \frac{f_i^n}{[n]_{q_i}!}.$$

**7.4.1.** Recall from 4.5.3 that the braid group of  $W$  associated to  $\mathfrak{g}$  acts as a group of automorphisms on  $\mathcal{U}_q(\mathfrak{g})$ . This result extends [Lus93] to an action of the braid group  $\tilde{B}$  associated to  $\tilde{W}$  on  $\mathcal{U}_q(\hat{\mathfrak{g}})$  as a group of automorphisms. The generators of  $\tilde{B}$  are  $\{T_i^{\pm 1} \mid i \in [0, r]\}$ . Let  $l$  be the length function on  $\tilde{W}$ . Let  $s_{i_1} s_{i_2} \cdots s_{i_n}$  ( $i_k \in [0, r]$ ) be a reduced expression of  $w \in \tilde{W}$  ( $l(w) = n$ ). Then define  $T_w := T_{i_1} T_{i_2} \cdots T_{i_n}$ . The map  $\tilde{W} \rightarrow \tilde{B}$ , that maps  $w \mapsto T_w$  such that  $T_{w_1} T_{w_2} = T_{w_1 w_2}$  if  $l(w_1) + l(w_2) = l(w_1 w_2)$ , is unique and well-defined. The action of the braid group generators on the generators of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  is given by

$$\begin{aligned} T_i &: k_i \mapsto k_i^{-1}, \\ T_i &: k_j \mapsto k_j k_i^{-a_{ij}} \quad (i \neq j), \\ T_i &: e_i \mapsto -k_i f_i, \\ T_i &: e_j \mapsto \sum_{n=0}^{-a_{ij}} (-1)^n q_i^{a_{ij}+n} e_i^{(n)} e_j e_i^{(-a_{ij}-n)} \quad (i \neq j), \\ T_i &: f_i \mapsto -k_i^{-1} e_i, \\ T_i &: f_j \mapsto \sum_{n=0}^{-a_{ij}} (-1)^n q_i^{-a_{ij}-n} f_i^{(-a_{ij}-n)} f_j f_i^{(n)} \quad (i \neq j). \end{aligned}$$

The action of the inverse elements is given by  $T_i^{-1} = \varphi \circ T_i \circ \varphi^{-1}$  ( $i \in [0, r]$ ). Note also that  $T_i \circ \omega = \omega \circ T_i$  ( $i \in [0, r]$ ). The braid group  $\tilde{B}$  and its action on  $\mathcal{U}_q(\hat{\mathfrak{g}})$  can be extended [Bec93] to include also the finite group  $\mathcal{T}$  with a natural action on  $\mathcal{U}_q(\hat{\mathfrak{g}})$ , so giving the group  $\hat{B}$  of automorphisms of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  associated to  $\hat{W} = \mathcal{T} \ltimes \tilde{W}$ . For  $\tau \in \mathcal{T}$ , define  $T_\tau$  which acts on  $\mathcal{U}_q(\hat{\mathfrak{g}})$  as

$$\begin{aligned} T_\tau(e_i) &= e_{\tau(i)}, & T_\tau(f_i) &= f_{\tau(i)}, \\ T_\tau(k_i) &= k_{\tau(i)}. \end{aligned}$$

The group  $\hat{B}$  is then generated by  $\{T_i^{\pm 1}, T_\tau \mid i \in [0, r], \tau \in \mathcal{T}\}$ .

**7.4.2. Lattice of translations.** The subgroup  $P^\vee$  of  $\hat{W}$ , induces a subgroup  $\mathcal{P}$  (the lattice of translations) of  $\hat{B}$ , whose generators are  $\{(T_{\omega_i^\vee})^{\pm 1} \mid i \in [1, r]\}$ .

REMARK 7.4.3. Note the action of  $\hat{B}$  on  $\mathcal{U}_q(\hat{\mathfrak{g}})$  is only known in terms of the generators  $\{T_i, T_\tau \mid i \in [0, r], \tau \in \mathcal{T}\}$  that correspond to the  $\mathcal{T} \ltimes \tilde{W}$  form of  $\hat{W}$ : calculating explicitly the bijection between  $W \ltimes P^\vee$  and  $\mathcal{T} \ltimes \tilde{W}$  is somewhat complicated. From

here on this bijection will be assumed and braid group generators  $\{T_i, T_{\omega_i^\vee} \mid i \in [1, r]\}$  corresponding to the  $W \ltimes P^\vee$  form of  $\hat{W}$  will be used.

NOTATION. Define an orientation (colouring) of the vertices of the Dynkin diagram of  $\mathfrak{g}$ , by a map  $o : [1, r] \rightarrow \{1, -1\}$ , such that if  $a_{ij} < 0$  then  $o(i)o(j) = -1$ . Then define  $\hat{T}_{\omega_i^\vee} := o(i)T_{\omega_i^\vee}$ .

### 7.5. Drinfeld's presentation

In [Dri88] Drinfeld introduced an important loop-like presentation of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  which I now describe. It turns out to be very useful for many calculations and applications of quantum affine Kac-Moody algebras.

DEFINITION 7.5.1. [Dri88] Let  $\mathfrak{g}$  be a simple, rank  $r$  Lie algebra, with Cartan matrix  $(a_{ij})_{i,j \in \{1, \dots, r\}}$ . The *centrally extended quantum loop algebra*  $\mathcal{U}_q(\widehat{L(\mathfrak{g})})$  of  $\mathfrak{g}$  with derivation is defined to be the associative  $\mathbb{C}(q)$ -algebra with unity 1 and generators  $\{E_m^{+,i}, E_m^{-,i}, H_n^i, K_i^{\pm 1}, \gamma^{\pm \frac{1}{2}}, D^{\pm 1} \mid i \in [1, r], m \in \mathbb{Z}, n \in \mathbb{Z}^\times\}$  ( $\gamma^{\pm \frac{1}{2}}$  is central) satisfying the relations:

$$\begin{aligned} \gamma \cdot \gamma^{-1} &= 1 = \gamma^{-1} \cdot \gamma, & D \cdot D^{-1} &= 1 = D^{-1} \cdot D, \\ K_i \cdot K_j &= K_j \cdot K_i, & K_i \cdot K_i^{-1} &= 1 = K_i^{-1} \cdot K_i, \\ D \cdot K_i &= K_i \cdot D, \\ K_i \cdot H_n^j &= H_n^j \cdot K_i, & D \cdot H_n^i \cdot D^{-1} &= q^n H_n^i, \\ K_i \cdot E_m^{\pm, j} \cdot K_i^{-1} &= q_i^{\pm a_{ij}} E_m^{\pm, j}, & D \cdot E_m^{\pm, i} \cdot D^{-1} &= q^m E_m^{\pm, i}, \end{aligned}$$

$$\begin{aligned} [H_m^i, H_n^j] &= \delta_{m+n, 0} \frac{[a_{ij}m]_{q_i}}{m} \cdot \frac{\gamma^m - \gamma^{-m}}{q_j - q_j^{-1}}, \\ [H_m^i, E_n^{\pm, j}] &= \pm \frac{[a_{ij}m]_{q_i}}{m} \gamma^{\mp \frac{|m|}{2}} E_{m+n}^{\pm, j}, \\ [E_m^{+, i}, E_n^{-, j}] &= \delta^{ij} \frac{\gamma^{\frac{1}{2}(m-n)} \psi_{m+n}^{+, i} - \gamma^{-\frac{1}{2}(m-n)} \psi_{m+n}^{-, i}}{q_i - q_i^{-1}}, \end{aligned}$$

$$E_{m+1}^{\pm, i} \cdot E_n^{\pm, j} - q_i^{\pm a_{ij}} E_n^{\pm, j} \cdot E_{m+1}^{\pm, i} = q_i^{\pm a_{ij}} E_m^{\pm, i} \cdot E_{n+1}^{\pm, j} - E_{n+1}^{\pm, j} \cdot E_m^{\pm, i},$$

$$\begin{aligned} \sum_{n=0}^{1-a_{ij}} \sum_{\sigma \in S_{1-a_{ij}}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_{q_i} E_{m_{\sigma(1)}}^{\pm, i} \cdots E_{m_{\sigma(n)}}^{\pm, i} \cdot E_p^{\pm, j} \cdot E_{m_{\sigma(n+1)}}^{\pm, i} \cdots E_{m_{\sigma(1-a_{ij})}}^{\pm, i} \\ = 0 \quad (i \neq j). \end{aligned}$$

The last two relations are the  $q$ -Serre relations. I call the former the quadratic Serre relations and the latter simply the Serre relations. The  $\{\psi_m^{+, i}, \psi_m^{-, i} \mid i \in [1, r], m \in \mathbb{Z}\}$

are defined by equating powers of  $z$  in the formal series:

$$\begin{aligned}\sum_{m \in \mathbb{Z}} \psi_m^{+,i} z^{-m} &:= K_i \exp \left( (q_i - q_i^{-1}) \sum_{n=1}^{\infty} H_n^i z^{-n} \right), \\ \sum_{m \in \mathbb{Z}} \psi_{-m}^{-,i} z^m &:= K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{n=1}^{\infty} H_{-n}^i z^n \right).\end{aligned}$$

LEMMA 7.5.2. *There is a family of  $\mathcal{U}_q(\mathfrak{g})$  subalgebras of  $\mathcal{U}_q(\widehat{L(\mathfrak{g})})$  parametrised by the elements  $m \in \mathbb{Z}$ . Let  $\{k_i^{\pm 1}, e_i, f_i\}$  be the standard generators of  $\mathcal{U}_q(\mathfrak{g})$ . The embedding of each subalgebra  $\mathcal{U}_q(\mathfrak{g}) \hookrightarrow \mathcal{U}_q(\widehat{L(\mathfrak{g})})$  is given by the  $\mathbb{C}(q)$ -algebra homomorphism:*

$$k_i \mapsto \gamma^m K_i, \quad e_i \mapsto E_m^{+,i}, \quad f_i \mapsto E_{-m}^{-,i}.$$

The ‘horizontal’ subalgebra corresponds to the case with  $m = 0$ .

PROOF. The homomorphism is easily checked explicitly.  $\square$

Recently Beck gave a constructive proof of the following theorem by Drinfeld.

THEOREM 7.5.3. [Dri88] *The quantum affine Kac-Moody algebra  $\mathcal{U}_q(\hat{\mathfrak{g}})$  extended by a central element  $C^{\frac{1}{2}}$  (satisfying  $C = \prod_{i=0}^r k_i^{a_i}$ ) is isomorphic as an algebra to the centrally extended quantum loop algebra  $\mathcal{U}_q(\widehat{L(\mathfrak{g})})$ .*

PROOF. The theorem is proved in [Bec93], where the isomorphism  $\mathcal{U}_q(\hat{\mathfrak{g}}) \xrightarrow{\sim} \mathcal{U}_q(\widehat{L(\mathfrak{g})})$  is shown to be:

$$\begin{aligned}\gamma^{\frac{1}{2}} &= C^{\frac{1}{2}}, & K_i &= k_i, & (i \in [1, r]) \\ E_m^{+,i} &= (\hat{T}_{\omega_i^\vee})^{-m} e_i, & E_m^{-,i} &= (\hat{T}_{\omega_i^\vee})^m f_i, & (m \in \mathbb{Z}) \\ D &= D, \\ \psi_0^{+,i} &= k_i, & \psi_0^{-,i} &= k_i^{-1}, & (i \in [1, r]) \\ \psi_m^{+,i} &= (q_i - q_i^{-1}) C^{\frac{m}{2}} [e_i, \hat{T}_{\omega_i^\vee}^m f_i] & (m \in \mathbb{Z}_{>0}), \\ \psi_{-m}^{-,i} &= \omega(\psi_m^{+,i}) = (q_i - q_i^{-1}) C^{-\frac{m}{2}} [f_i, \hat{T}_{\omega_i^\vee}^m e_i] & (m \in \mathbb{Z}_{>0}), \\ H_m^i &= k_i^{-1} \left( \frac{\psi_m^{+,i}}{q_i - q_i^{-1}} - \frac{1}{m} \sum_{p=1}^{m-1} p \psi_{m-p}^{+,i} H_p^i \right) & (m \in \mathbb{Z}_{>0}), \\ H_{-m}^i &= \omega(H_m^i) & (m \in \mathbb{Z}_{>0}).\end{aligned}$$

The elements in  $\mathcal{U}_q(\widehat{L(\mathfrak{g})})$  on the left-hand side of each equation are the image under the map of the elements in  $\mathcal{U}_q(\mathfrak{g})$  on the right-hand side.  $\square$

COROLLARY 7.5.4. *The derived quantum Kac-Moody algebra  $\mathcal{U}_q(\hat{\mathfrak{g}}')$  is isomorphic to  $\mathcal{U}_q(\widehat{L(\mathfrak{g})}')$ , the subalgebra of  $\mathcal{U}_q(\widehat{L(\mathfrak{g})})$  without the derivation  $D$ .*

The elements  $H_m^i$  and  $\psi_m^{\pm,i}$  are fixed under the action of the translations  $\hat{T}_{\omega_i^\vee}$  and  $\gamma \hat{T}_{\omega_i^\vee}$  respectively (see [Dam93, §4]):



COROLLARY 7.5.5. *Let  $m \in \mathbb{Z}_{>0}$ . Then*

$$\begin{aligned}\hat{T}_{\omega_i^\vee} \psi_m^{+,i} &= \gamma^{-1} \psi_m^{+,i}, \\ \hat{T}_{\omega_i^\vee} H_m^i &= H_m^i.\end{aligned}$$

REMARK 7.5.6. In Drinfeld's presentation of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ , the Cartan subalgebra  $\mathcal{U}_q(\hat{\mathfrak{h}})$  is generated by  $\{K_i^{\pm 1}, \gamma^{\pm \frac{1}{2}}, D^{\pm 1}\}$ . The set  $\{E_0^{+,i}, E_m^{\pm,i}, H_m^i \mid m \in \mathbb{Z}_{>0}\}$  generates  $\mathcal{U}_q(\hat{\mathfrak{n}}_+)$  and  $\{E_0^{-,i}, E_m^{\pm,i}, H_m^i \mid m \in \mathbb{Z}_{<0}\}$  generates  $\mathcal{U}_q(\hat{\mathfrak{n}}_-)$ .

**7.5.7. Basis.** For fixed  $m \in \mathbb{Z}^\times$ , the generators  $\{H_m^i \mid i \in [1, r]\}$  are the imaginary root vectors associated to the imaginary root  $m\delta$ . The generators  $\{H_m^i \mid i \in [1, r], m \in \mathbb{Z}_{>0}\}$  are the positive imaginary root vectors and  $\{H_{-m}^i \mid i \in [1, r], m \in \mathbb{Z}_{>0}\}$  the negative imaginary root vectors. The generator  $E_m^{\pm,i}$  corresponds to the real root  $\pm\alpha_i + m\delta$ .

Let  $s_{i_1} s_{i_2} \dots s_{i_n}$  be a reduced expression of the longest element  $w_0 \in W$ . Then the ordered set  $\{\beta_k := s_{i_1} s_{i_2} \dots s_{i_{k-1}} \alpha_{i_k} \mid k \in [1, n]\}$  is  $R_+$  with an ordering. For each  $m \in \mathbb{Z}$ , I define

$$\begin{aligned}E_m^{+, \beta_k} &:= T_{i_1} T_{i_2} \dots T_{i_{k-1}} E_m^{+, i_k}, \\ E_m^{-, \beta_k} &:= \omega(E_m^{+, \beta_k}).\end{aligned}$$

Then the set  $\{E_0^{+, \beta_k}, E_m^{\pm, \beta_k}, H_m^i \mid m \in \mathbb{Z}_{>0}, k \in [1, n]\}$  generates a basis of  $\mathcal{U}_q(\hat{\mathfrak{n}}_+)$  in an appropriate order (such as the one in 7.3.6). Similarly  $\{E_0^{-, \beta_k}, E_m^{\pm, \beta_k}, H_m^i \mid m \in \mathbb{Z}_{<0}, k \in [1, n]\}$  generates a basis of  $\mathcal{U}_q(\hat{\mathfrak{n}}_-)$ .

The basis of  $\mathcal{U}_q(\hat{\mathfrak{h}})$  is simply

$$\mathcal{U}_q(\hat{\mathfrak{h}}) = \sum_{m_i, m_c, m_d \in \mathbb{Z}} K_1^{m_1} \dots K_r^{m_r} \gamma^{\frac{m_c}{2}} D^{m_d}.$$

**7.5.8.** The Hopf algebra structure of  $\mathcal{U}_q(\widehat{L(\mathfrak{g})})$  is not known explicitly in terms of Drinfeld's generators. However Chari and Pressley have given some partial results for the coproduct of  $\mathcal{U}_q(L(\mathfrak{sl}_2))$  [CP91b]. Beck recently presented a construction of the coproduct for the real root vectors  $\{E_m^{\pm,i} \mid m \in \mathbb{Z}\}$  in  $\mathcal{U}_q(\widehat{L(\mathfrak{g})})$  using the  $\mathcal{R}$ -matrix of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ . From this the coproduct for the imaginary root vectors can also be deduced.

**7.5.9.** The set  $\{K_i, H_m^i, \gamma^{\pm \frac{1}{2}} \mid m \in \mathbb{Z}\}_{i \in [1, r]}$  generates a subalgebra  $\mathcal{U}_q(\mathfrak{H})$  of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  called the *Heisenberg subalgebra*. If I adjoin the formal central elements  $\ln(q)$  and  $\ln(q)^{-1}$  to the algebra, then the generators  $\{H_m^i, K_i, x^i \mid i \in [1, r]\}$  ( $x^i$  conjugate to the zero mode momenta  $K_i$ :  $[K_i, x^j] = \ln(q) \delta^{ij} K_i$ ) are Fourier modes of  $q$ -analogue 2d chiral bosonic quantum fields  $\{\phi^i(z)\}$ :

$$\begin{aligned}\phi^i(z) &= x^i + \frac{1}{2}(K_i - K_i^{-1}) \frac{\ln(z)}{\ln(q_i)} + \sum_{n \in \mathbb{Z}^\times} \frac{\gamma^{\frac{1}{2}|n|}}{[n]_{q_i}} H_n^i z^{-n} \\ \partial_{q_i} \phi^i(z) &= \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} z^{-1} + \sum_{n \in \mathbb{Z}^\times} \gamma^{\frac{1}{2}|n|} H_n^i z^{-n-1}.\end{aligned}$$

Here the finite difference operator acts as  $\partial_q f(z) := \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}$  on any complex function  $f(z)$ . These  $q$ -bosonic fields play a crucial role in the constructions of bosonisations of quantum affine Kac-Moody algebras and their vertex operators (see the references cited in 1.3.1 and references therein).

REMARK 7.5.10. Setting  $\gamma^{\frac{1}{2}}$  to 1 in  $\mathcal{U}_q(L(\mathfrak{g}))$ , corresponds to having zero central extension in the classical limit. I define the quantum loop algebra of  $\mathfrak{g}$  (with zero central extension) to be  $\mathcal{U}_q(L(\mathfrak{g})) := \mathcal{U}_q(\widehat{L(\mathfrak{g})}) / \langle \gamma^{\frac{1}{2}} - 1 \rangle$ . The subalgebra  $\mathcal{U}_q(\mathfrak{H}) / \langle \gamma^{\frac{1}{2}} - 1 \rangle$  of  $\mathcal{U}_q(L(\mathfrak{g}))$  forms a commutative subalgebra. So the Cartan subalgebra of  $\mathcal{U}_q(L(\mathfrak{g}))$  is generated by  $\{K_i^{\pm 1}, D^{\pm 1}\} \cup \mathcal{U}_q(\mathfrak{H}) / \langle \gamma^{\frac{1}{2}} - 1 \rangle$ .

LEMMA 7.5.11. *Let  $m, n \in \mathbb{Z}$  be two integers such that  $m > n$ .*

(a) *If  $m = n + 1$*

$$E_m^{\pm, i} E_n^{\pm, i} - q_i^{\pm 2} E_n^{\pm, i} E_m^{\pm, i} = 0.$$

(b) *For  $i, j \in [1, r]$*

$$\begin{aligned} E_m^{\pm, i} E_n^{\pm, j} - q_i^{\pm a_{ij}} E_n^{\pm, j} E_m^{\pm, i} &= (q_i^{\pm 2a_{ij}} - 1) \sum_{p=1}^{\ell} q_i^{\pm(p-1)a_{ij}} E_{n+p}^{\pm, j} E_{m-p}^{\pm, i} \\ &\quad + q_i^{\pm \ell a_{ij}} (q_i^{\pm a_{ij}} E_{m-\ell-1}^{\pm, i} E_{n+\ell+1}^{\pm, j} - E_{n+\ell+1}^{\pm, j} E_{m-\ell-1}^{\pm, i}). \end{aligned}$$

(c) *If  $m > n + 2$  and  $m - n$  is odd, then*

$$E_m^{\pm, i} E_n^{\pm, i} - q_i^{\pm 2} E_n^{\pm, i} E_m^{\pm, i} = (q_i^{\pm 4} - 1) \sum_{j=1}^{m-n-1} q_i^{\pm 2(j-1)} E_{n+j}^{\pm, i} E_{m-j}^{\pm, i}.$$

(d) *If  $m > n + 1$  and  $m - n$  is even,*

$$\begin{aligned} E_m^{\pm, i} E_n^{\pm, i} - q_i^{\pm 2} E_n^{\pm, i} E_m^{\pm, i} &= (q_i^{\pm 4} - 1) \sum_{j=1}^{m-n-2} q_i^{\pm 2(j-1)} E_{n+j}^{\pm, i} E_{m-j}^{\pm, i} \\ &\quad + (q_i^{\pm 2} - 1) q_i^{m-n-2} E_{\frac{m+n}{2}}^{\pm, i} E_{\frac{m+n}{2}}^{\pm, i}. \end{aligned}$$

PROOF. (a) The  $m = n + 1$  case follows immediately from the quadratic Serre relations. The proof of (b) is by induction for fixed  $n$  using the quadratic Serre relations recursively. (c) and (d) are special cases of (b).  $\square$

REMARK 7.5.12. Note that in the case of  $\mathcal{U}_q(\widehat{L(\mathfrak{sl}_2)})$  there are no Serre relations just as in the classical case, but there are the quadratic Serre relations. So lemma 7.5.11 allows us to order the sets of root vectors  $\{E_m^+\}_{m \in \mathbb{Z}}$  and  $\{E_m^-\}_{m \in \mathbb{Z}}$  in  $\mathcal{U}_q(\widehat{L(\mathfrak{sl}_2)})$  and within the  $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$  subalgebras of  $\mathcal{U}_q(\widehat{\mathfrak{g}})$ . For  $i, j$  such that  $a_{ij} = 0$ , the Serre relation tells us that  $E_m^{\pm, i}$  and  $E_n^{\pm, j}$  commute for all  $m, n \in \mathbb{Z}$  as for a classical affine Kac-Moody algebra, so the quadratic Serre relation is trivially satisfied. On the other hand for  $i, j$  such that  $a_{ij} < 0$ , the quadratic Serre relation gives additional relations to the Serre relation.

**7.5.13.** Let  $\{\bar{H}_m^i, \bar{E}_m^{\pm,i}, \bar{k}, \bar{d} \mid m \in \mathbb{Z}\}$  denote the usual loop algebra presentation of  $\mathcal{U}(\hat{\mathfrak{g}})$  with central element  $\bar{k}$  and derivation  $\bar{d}$ . The generators satisfy the relations:

$$\begin{aligned} [\bar{H}_m^i, \bar{H}_n^j] &= \delta_{m+n} a_{ij} m \bar{k}, \\ [\bar{H}_m^i, \bar{E}_n^{\pm,j}] &= \pm a_{ij} \bar{E}_{m+n}^{\pm,j}, \\ [\bar{E}_m^{+,i}, \bar{E}_n^{-,j}] &= \delta^{ij} (\bar{H}_{m+n}^i + m \bar{k}), \\ [\bar{E}_m^{\pm,i}, \bar{E}_n^{\pm,i}] &= 0, \\ \left[ E_{m_1}^{\pm,i}, \left[ E_{m_2}^{\pm,i}, \left[ \dots, \left[ E_{m_1-a_{ij}}^{\pm,i}, E_n^{\pm,j} \right] \right] \dots \right] \right] &= 0 \quad (i \neq j). \end{aligned}$$

LEMMA 7.5.14 (CLASSICAL LIMIT). Define  $\mathcal{U}_A(\widehat{L(\mathfrak{g})})$  to be the  $\mathbb{C}[q, q^{-1}]$ -subalgebra of  $\mathcal{U}_q(\widehat{L(\mathfrak{g})})$  generated by

$$H_0^i := \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad d := \frac{D - D^{-1}}{q - q^{-1}}, \quad k := \frac{\gamma - \gamma^{-1}}{q - q^{-1}},$$

and all the generators of  $\mathcal{U}_q(\widehat{L(\mathfrak{g})})$ . Let  $\epsilon \in \mathbb{C}^\times$ . Define the specialisation of  $\mathcal{U}_A(\widehat{L(\mathfrak{g})})$  at  $q = \epsilon$  to be  $\mathcal{U}_\epsilon(\widehat{L(\mathfrak{g})}) := \mathcal{U}_A(\widehat{L(\mathfrak{g})}) / \langle q - \epsilon \rangle$ . The quotient algebra  $\mathcal{U}_1(\hat{\mathfrak{g}}) / \langle D - 1, \gamma^{\frac{1}{2}} - 1, K_i - 1 \mid i \in [1, r] \rangle$  is  $\mathbb{C}$ -algebra isomorphic to  $\mathcal{U}(\hat{\mathfrak{g}})$ :

$$\begin{aligned} H_m^i &\mapsto \bar{H}_m^i, & E_m^{\pm,i} &\mapsto \bar{E}_m^{\pm,i}, & (m \in \mathbb{Z}) \\ k &\mapsto \bar{k}, & d &\mapsto \bar{d}. \end{aligned}$$

REMARK 7.5.15. From lemma 7.5.11, it follows that the quadratic Serre relations with  $i = j$  (and in particular for  $\mathcal{U}_q(\widehat{L(\mathfrak{sl}_2)})$ ) become trivial at the specialisation  $q = 1$ , as would be expected for consistency. The quadratic Serre relations have no classical analogue with  $i = j$ :  $[\bar{E}_m^{\pm,i}, \bar{E}_n^{\pm,i}] = 0$  in  $\mathcal{U}(\widehat{L(\mathfrak{g})})$ .

**7.5.16.** In Drinfeld's presentation the  $\mathbb{C}$ -algebra anti-automorphism  $\omega$  (Cartan involution) takes the form

$$\begin{aligned} K_i &\mapsto K_i^{-1}, & q &\mapsto q^{-1}, \\ \gamma^{\frac{1}{2}} &\mapsto \gamma^{-\frac{1}{2}}, & D &\mapsto D^{-1}, \\ E_m^{\pm,i} &\mapsto E_{-m}^{\mp,i}, & H_m^i &\mapsto H_{-m}^i. \end{aligned}$$

DEFINITION 7.5.17. There is a natural homogeneous  $\mathbb{Z}$ -gradation of  $\mathcal{U}_q(\widehat{L(\mathfrak{g})})$  given by:

$$\begin{aligned} \deg K_i &= 0, & \deg \gamma^{\frac{1}{2}} &= 0, & \deg D &= 0, \\ \deg H_m^i &= z^m, & \deg E_m^{\pm,i} &= z^m. \end{aligned}$$

**7.5.18. The evaluation map.** Let  $n > 1$  and let  $\hat{\mathfrak{g}}$  be a affine Lie algebra of type  $A_{n-1}^{(1)} \simeq \widehat{\mathfrak{sl}}_n$ . The evaluation map of the quantum affine Kac-Moody algebra  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_n) \rightarrow \mathcal{U}_q(\mathfrak{sl}_n) \otimes \mathbb{C}[z, z^{-1}]$  was first constructed by Jimbo [Jim86]. Chari and Pressley [CP91b] studied the evaluation representations (corresponding to the homogeneous gradation) of  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$  in detail. Apparently it is not possible to construct evaluation representations of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ , for  $\hat{\mathfrak{g}}$  of affine type other than  $A_{n-1}^{(1)}$ .

The evaluation map  $\text{ev}_z$  is useful because it allows one to construct loop modules of  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_n')$  over  $\mathcal{U}_q(\mathfrak{sl}_n)$ -modules and also to construct parameter dependent solutions of the Yang-Baxter equation (see also [BDGZ93]). When the formal parameter  $z$  is taken to be a non-zero complex number  $a$  then  $\text{ev}_a : \mathcal{U}_q(\widehat{\mathfrak{sl}}_n') \rightarrow \mathcal{U}_q(\mathfrak{sl}_n)$  and irreducible finite dimensional representations of  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_n')$  can be obtained. Note that an evaluation map of  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_n')$  onto a finite dimensional highest weight  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_n)$ -module is not in general a highest weight representation of  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_n')$ , since  $e_0$  and  $f_0$  are mapped into the  $\mathcal{U}_q(\mathfrak{n}_-)$  and  $\mathcal{U}_q(\mathfrak{n}_+)$  subalgebras of  $\mathcal{U}_q(\mathfrak{sl}_n)$  respectively. The evaluation map is *not* a Hopf algebra map, so in particular the tensor product structure of  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_n)$ -loop modules is different from that of the corresponding  $\mathcal{U}_q(\mathfrak{sl}_n)$ -modules. The evaluation representations of  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2')$  have been classified by Chari and Pressley [CP91b].

REMARK 7.5.19. Recently an interesting two parameter (elliptic) deformation of  $\widehat{\mathfrak{sl}}_2$  was introduced [FIJKM<sup>+</sup>94].

## 7.6. Heisenberg subalgebra

In this section I consider the Heisenberg subalgebra  $\mathcal{U}_q(\mathfrak{H})$  of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ , which has the generators  $\{H_n^i, K_i^{\pm 1}, \gamma^{\pm \frac{1}{2}} \mid n \in \mathbb{Z}^\times\}_{i \in [1, r]}$ . The generators  $K_i^{\pm 1}$  and  $\gamma^{\pm \frac{1}{2}}$  are central in  $\mathcal{U}_q(\mathfrak{H})$  and

$$[H_m^i, H_n^j] = \delta_{m+n, 0} \frac{[a_{ij}m]_{q_i}}{m} \cdot \frac{\gamma^m - \gamma^{-m}}{q_j - q_j^{-1}} \quad (m, n \in \mathbb{Z}^\times; i, j \in [1, r]).$$

Let  $\mathcal{U}_q(\mathfrak{H}^+)$  and  $\mathcal{U}_q(\mathfrak{H}^-)$  denote the commutative subalgebras of  $\mathcal{U}_q(\mathfrak{H})$  generated by  $\{H_n^i \mid n \in \mathbb{Z}_{>0}\}$  and  $\{H_n^i \mid n \in \mathbb{Z}_{<0}\}$  respectively.

REMARK 7.6.1. The Heisenberg algebra of  $\mathcal{U}_q(\mathfrak{H})$  of  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$  is essentially an infinite dimensional generalisation of the Heisenberg subalgebra of the  $q$ -oscillator algebra mentioned in 5.3.3.

**7.6.2.** Let  $\mathcal{U}_A(\mathfrak{H})$  denote the  $\mathbb{C}[q, q^{-1}]$ -subalgebra of  $\mathcal{U}_q(\mathfrak{H})$  generated by  $\{\gamma^{\pm \frac{1}{2}}, k := \frac{\gamma - \gamma^{-1}}{q - q^{-1}}, H_0^i := \frac{K_i - K_i^{-1}}{q - q^{-1}}, H_m^i \mid m \in \mathbb{Z}, i \in [1, r]\}$ . Let  $\epsilon \in \mathbb{C}^\times$ . The specialisation of  $\mathcal{U}_A(\mathfrak{H})$  at  $q = \epsilon$  is  $\mathcal{U}_\epsilon(\mathfrak{H}) := \mathcal{U}_A(\mathfrak{H}) / \langle q - \epsilon \rangle$ . Let  $\mathcal{U}(\mathfrak{H})$  denote the Heisenberg enveloping subalgebra of  $\mathcal{U}(\hat{\mathfrak{g}})$  introduced in 7.5.13, generated by  $\{\bar{k}, \bar{H}_m^i \mid m \in \mathbb{Z}\}$ . From 7.5.14 it follows that  $\mathcal{U}_1(\mathfrak{H}) / \langle \gamma - 1, K_i - 1 \mid i \in [1, r] \rangle \simeq \mathcal{U}(\mathfrak{H})$ . The isomorphism is given by  $H_m^i \mapsto \bar{H}_m^i$  and  $k \mapsto \bar{k}$ .

**7.6.3.** Let  $F$  be a Fock module over  $\mathbb{C}(q)$  of  $\mathcal{U}_q(\mathfrak{H})$ , generated by an element  $v_0 \in F$ , called the vacuum vector, such that  $\mathcal{U}_q(\mathfrak{H}^+) \cdot v_0 = 0$  and  $F = \mathcal{U}_q(\mathfrak{H}^-) \cdot v_0$ .

LEMMA 7.6.4 (IRREDUCIBILITY).  $F$  is an irreducible representation of  $\mathcal{U}_q(\mathfrak{H})$ , if and only if  $\gamma^2 \cdot v \neq v$  ( $v \in F$ ).

PROOF. The proof is similar to the classical case [KR87, 2.2]. From any element of  $F$  one can reach the vacuum vector  $v_0$  by appropriate applications of elements of  $\mathcal{U}_q(\mathfrak{H}^+)$ . Then acting with elements of  $\mathcal{U}_q(\mathfrak{H}^-)$  any other element of  $F$  can be reached. If  $\gamma^2 \cdot v = v$  ( $v \in F$ ), then clearly  $[H_m^i, H_n^j]$  acts on  $F$  as 0, which would give a trivial representation.  $\square$

NOTATION. A nonzero complex number is called *generic*, if it is real or if it is not equal to a (nontrivial) root of unity.

LEMMA 7.6.5. Let  $F$  be a  $\mathbb{C}(q)$ -Fock module of  $\mathcal{U}_q(\mathfrak{H})$  with vacuum vector  $v_0$ .  $F_{\mathcal{A}} := \mathcal{U}_{\mathcal{A}}(\mathfrak{H}) \cdot v_0$  is a  $\mathcal{U}_{\mathcal{A}}(\mathfrak{H})$  submodule of  $F$ . If  $F$  is an irreducible Fock module of  $\mathcal{U}_q(\mathfrak{H})$ , then the Fock module  $F_{\mathcal{A}}$  over  $\mathcal{U}_{\mathcal{A}}(\mathfrak{H})$  is also irreducible.

Let  $\epsilon \in \mathbb{C}^\times$ . Then  $F_\epsilon := F_{\mathcal{A}} / \langle q - \epsilon \rangle$  is a  $\mathcal{U}_\epsilon(\mathfrak{H})$  submodule of  $F_{\mathcal{A}}$ . If  $F_{\mathcal{A}}$  is an irreducible Fock module of  $\mathcal{U}_{\mathcal{A}}(\mathfrak{H})$ , then its specialisation  $F_\epsilon$  over  $\mathcal{U}_\epsilon(\mathfrak{H})$  at generic  $q = \epsilon$  is also irreducible.

**7.6.6.** Let  $F$  be an irreducible Fock module of  $\mathcal{U}_q(\mathfrak{H})$ . When the central elements  $\gamma$  and  $K_i$  ( $i \in [1, r]$ ) of  $\mathcal{U}_q(\mathfrak{H})$  act on  $F$  with eigenvalues  $q^c$  ( $c \in \mathbb{Z}$ ) and  $q^{\alpha_i}$  ( $\alpha_i \in \mathbb{Z}$ ,  $i \in [1, r]$ ), I call  $c$  the *level* and  $\alpha := (\alpha_i)_{i \in [1, r]}$  the *charge* (or momentum) respectively of the Fock module  $F$ . Write  $F_\alpha$  for  $F$ , when it is necessary to emphasise the charge of  $F$ .

In the following I will only consider (non-trivial) representations with non-zero level  $c \in \mathbb{Z}^\times$ .

LEMMA 7.6.7 (INTERTWINERS). Let  $x^i$  ( $i \in [1, r]$ ) be the zero mode coordinate conjugate to  $K_i$  introduced in 7.5.9. Let  $F_\alpha$  and  $F_\beta$  be two level  $c$  Fock modules over  $\mathcal{U}_q(\mathfrak{H})$  with charge  $\alpha$  and  $\beta$ . Then the map  $e^{\sum_{i=1}^r (\beta_i - \alpha_i)x^i} : F_\alpha \rightarrow F_\beta$  is a  $\mathcal{U}_q(\mathfrak{H})$ -module intertwiner from  $F_\alpha$  to  $F_\beta$ .

PROOF. Follows straightforwardly from  $K_i \cdot e^{x^j} = q^{\delta^{ij}} e^{x^j} \cdot K_i$  and  $[H_n^i, x^j] = 0$ .  $\square$

**7.6.8.** Recall from 7.5.16 that the Cartan involution  $\omega$  of  $\mathcal{U}_q(\mathfrak{H})$  is

$$\begin{aligned} q &\mapsto q^{-1}, & H_n^i &\mapsto H_{-n}^i, \\ K_i &\mapsto K_i^{-1}, & \gamma &\mapsto \gamma^{-1}, \end{aligned}$$

which extends as an anti-automorphism to all of  $\mathcal{U}_q(\mathfrak{H})$ . Note that the elements  $\frac{K_i - K_i^{-1}}{q - q^{-1}}$  and  $\frac{\gamma - \gamma^{-1}}{q - q^{-1}}$  in  $\mathcal{U}_q(\mathfrak{H})$  are invariant under  $\omega$ . Hence the elements  $H_0^i$  and  $k$  in  $\mathcal{U}_{\mathcal{A}}(\mathfrak{H})$  and  $\mathcal{U}_\epsilon(\mathfrak{H})$  are  $\omega$  invariant. Clearly  $\omega$  is also an anti-automorphism of  $\mathcal{U}_{\mathcal{A}}(\mathfrak{H})$  and  $\mathcal{U}_\epsilon(\mathfrak{H})$ .

**7.6.9.** Let  $(\cdot, \cdot) : F \times F \rightarrow \mathbb{C}(q)$  denote the unique scalar product on  $F$ , which is contravariant with respect to  $\omega$ :

$$\begin{aligned} (x \cdot v, w) &= (v, \omega(x) \cdot w), \\ (v, x \cdot w) &= (\omega(x) \cdot v, w), \end{aligned} \quad \forall x \in \mathfrak{H}, v, w \in F$$

and normalised so that  $(v_0, v_0) := 1$ . Denote by  $(\cdot, \cdot)_{\mathcal{A}}$  and  $(\cdot, \cdot)_{\epsilon}$  the induced scalar products on  $F_{\mathcal{A}}$  and  $F_{\epsilon}$ .

LEMMA 7.6.10. *The triple  $(F, (\cdot, \cdot), \omega)$  is a  $\star$ -representation of  $\mathcal{U}_q(\mathfrak{H})$ . The triple  $(F_{\mathcal{A}}, (\cdot, \cdot)_{\mathcal{A}}, \omega)$  is a  $\star$ -representation of  $\mathcal{U}_{\mathcal{A}}(\mathfrak{H})$ . Let  $\epsilon \in \mathbb{C}^{\times}$ . The triple  $(F_{\epsilon}, (\cdot, \cdot), \omega)$  is a  $\star$ -representation of  $\mathcal{U}_{\epsilon}(\mathfrak{H})$ . If  $\epsilon \in \mathbb{R}_{>0}$  then  $(F_{\epsilon}, (\cdot, \cdot), \omega)$  is a unitary representation of  $\mathcal{U}_{\epsilon}(\mathfrak{H})$ .*

PROOF. For  $\epsilon$  positive, it is easily checked that the sesquilinear scalar product is positive definite. Hence in this case the representation is unitary.  $\square$

## CHAPTER 8

### Quantum affine algebras at a root of unity

#### 8.1. Introduction

**8.1.1.** In this chapter the specialisation of the quantum affine Kac-Moody algebra  $\mathcal{U}_A(\hat{\mathfrak{g}})$  at an odd primitive root of unity is studied. Using the action of Beck's extended braid group  $\hat{B}$  introduced in the previous chapter on some well-known central elements in  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ , I construct an infinite number of central elements in  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  lying in  $\mathcal{U}_\epsilon(\hat{\mathfrak{n}}_\pm)$ . In fact I prove that, at an odd  $\ell$ -th root of unity, the  $\ell$ -th power of every real root vector in  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  is in the centre (proposition 8.3.5).

The Heisenberg algebra  $\mathcal{U}_\epsilon(\mathfrak{H})$  is also studied at odd and even roots of unity. It is found that it also contains an infinite number of central elements: the generators of the extended centre of  $\mathcal{U}_\epsilon(\mathfrak{H})$  are in fact the generators that have mode number, which is an integer (half integer in the even case) multiple of  $\ell$ . Further it turns out that the extended central elements of  $\mathcal{U}_\epsilon(\mathfrak{H})$  are also in the centre of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ .

The centre  $\mathcal{Z}_\epsilon$  of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  is infinite dimensional. Nevertheless  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  is still infinite dimensional over  $\mathcal{Z}_\epsilon$ , in contrast with the finite case.

**8.1.2.** Until now, quantum affine algebras at a root of unity have only been studied with zero central extension. In particular their (finite dimensional) minimal cyclic representations has been constructed [DJMM90, DJMM91b, AC91a, CP91a]. In this chapter infinite dimensional representations of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  and  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  are also discussed. The results on the representation theory are not definitive, but I am able to construct some new representations of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ , which are infinite dimensional and either nilpotent or semicyclic. It is also possible to construct cyclic representations, on which all the real root vectors act injectively.

The Fock modules over the Heisenberg subalgebra  $\mathcal{U}_\epsilon(\mathfrak{H})$  become reducible at the root of unity because of the new central elements. However it is possible to quotient by a maximal submodule generated by all the singular vectors giving an irreducible representation. An alternative approach is to take a Lusztig form of  $\mathcal{U}_A(\mathfrak{H})$ , in this case at the specialisation at a root of unity the algebra is still isomorphic to the classical Heisenberg subalgebra.

**8.1.3. Historical note.** Let me mention how I came upon the results concerning the central elements generated by the real root vectors in the algebra. Initially in the

Summer of 1993, I started by considering Drinfeld's presentation of  $\mathcal{U}_q(\mathfrak{sl}_2)$  and made a conjecture that the generators corresponding to real root vectors to the  $\ell$ -th power were central. For a long time I could only check the conjecture for generators with small mode number by rather long tedious calculations and with help from the symbolic manipulation programming language FORM [Ver89]. Then at the end of April 1994 I received the paper of Beck that proved the isomorphism between the Drinfeld and Chevalley type presentations of  $\mathcal{U}_q(\hat{\mathfrak{g}})$ . After I had read this paper, I then realised that I could apply his extension of the braid group action on  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  to prove my conjecture for general  $\hat{\mathfrak{g}}$ .

### 8.2. Heisenberg subalgebra at a root of unity

**8.2.1.** Let  $\ell$  be an integer such that  $\ell > 2$  and  $\ell > d_i$  ( $i \in [0, r]$ ) and fix  $\epsilon$  to be an primitive  $\ell$ -th root of unity ( $\epsilon^\ell = 1$ ). As usual define

$$\ell' = \begin{cases} \ell & \text{if } \ell \text{ is odd,} \\ \frac{\ell}{2} & \text{if } \ell \text{ is even.} \end{cases}$$

**PROPOSITION 8.2.2 (CENTRE).** *The elements  $\{H_{m\ell'}^i \mid m \in \mathbb{Z}^\times, i \in [1, r]\}$  are central in  $\mathcal{U}_\epsilon(\mathfrak{H})$  at the root of unity  $\epsilon$ . In fact these elements are also central in  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  at the root of unity.*

**PROOF.** This follows immediately from the commutation relations in 7.5.1, since  $[m\ell']_\epsilon = 0$  for all  $m \in \mathbb{Z}$ .  $\square$

**8.2.3.** So at an  $\ell$ -th root of unity the centre of the Heisenberg algebra  $\mathcal{U}_\epsilon(\mathfrak{H})$  is infinite dimensional. It is generated by  $\{\gamma^{\pm\frac{1}{2}}, K_i^{\pm 1}, H_{m\ell'}^i \mid m \in \mathbb{Z}^\times, i \in [1, r]\}$ . The generators  $H_m^i$ , with mode number  $m$  equaling a multiple of  $\ell'$ , 'decouple' into the centre of the algebra.

**COROLLARY 8.2.4.** *The Fock module  $F_\epsilon$  of  $\mathcal{U}_\epsilon(\mathfrak{H})$  at the root of unity  $\epsilon$  is reducible.*

**PROOF.** This is clear since the central elements  $H_{-m\ell'}^i$  ( $m \in \mathbb{Z}_{>0}, i \in [1, r]$ ) generate an infinity of singular vectors in  $F_\epsilon$ .  $\square$

**8.2.5.** At the root of unity I define a new triangular decomposition of  $\mathcal{U}_\epsilon(\mathfrak{H})$ . Let  $\mathcal{U}_\epsilon(\mathfrak{H}'_+)$ ,  $\mathcal{U}_\epsilon(\mathfrak{H}'_0)$  and  $\mathcal{U}_\epsilon(\mathfrak{H}'_-)$  be the commutative algebras generated respectively by  $\{H_m^i \mid m \in \mathbb{Z}_{>0} \setminus \ell'\mathbb{Z}_{>0}\}$ ,  $\{\gamma^{\pm\frac{1}{2}}, K_i^{\pm 1}, H_n^i \mid n \in \ell'\mathbb{Z}^\times\}$  and  $\{H_m^i \mid m \in \mathbb{Z}_{<0} \setminus \ell'\mathbb{Z}_{<0}\}$ .

**LEMMA 8.2.6.** *Define  $F'_\epsilon$  to be a  $\mathcal{U}_\epsilon(\mathfrak{H})$ -module at the root of unity  $\epsilon$  generated by a vector  $v'_0$ , such that  $\mathcal{U}_\epsilon(\mathfrak{H}'_+) \cdot v'_0 = 0$ ,  $F'_\epsilon = \mathcal{U}_\epsilon(\mathfrak{H}'_-) \cdot v'_0$ , and  $\gamma \cdot v'_0 = \epsilon^c v'_0$  ( $c \in \mathbb{C}^\times$ ). If  $c \notin \ell'\mathbb{Z}$ , then  $F'_\epsilon$  is an irreducible  $\mathcal{U}_\epsilon(\mathfrak{H})$ -module.*

**PROOF.** At level  $c$ ,  $[H_m^i, H_n^j]$  acts on  $F'_\epsilon$  as

$$(1) \quad [H_m^i, H_n^j] = \delta_{m+n,0} \frac{[a_{ij}m]_{\epsilon_i}}{m} \cdot [mc]_{\epsilon_j}.$$

The module is a highest weight module. If  $c \in \ell'\mathbb{Z}$ , then the right-hand side of (1) is always zero. Therefore every element of  $F'_\epsilon$  is singular and the module is reducible.



For  $c \notin \ell'\mathbb{Z}$ , if  $i, j \in [1, r]$  are such that  $a_{ij} \neq 0$ , then  $[H_m^i, H_{-m}^j]$  ( $m \in \mathbb{Z}^\times$ ) acts as a nonzero number on  $F'_\epsilon$  and the module  $F'_\epsilon$  is irreducible, since it contains no singular vectors.  $\square$

REMARK 8.2.7. Consider the  $q_i$ -boson field  $\phi^i(z)$  (defined in 7.5.9). It is interesting to note that the terms in  $\phi^i(z)$  that diverge at the root of unity  $\epsilon$  are exactly those in the central elements  $H_{m\ell'}^i \in \mathcal{U}_\epsilon(\tilde{\mathfrak{H}}'_0)$  (diverging by a factor  $\frac{1}{[m\ell']_{\epsilon_i}}$ ). This fact may be useful for the construction the vertex operators and bosonisation of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  at a root of unity.

**8.2.8.** Fix  $c \in \mathbb{Z}$ . Consider the algebra  $\mathcal{U}_q(\tilde{\mathfrak{H}}) := \mathcal{U}_q(\mathfrak{H})/\langle \gamma - q^c \rangle$  with generators  $\tilde{K}^i := K_i$  (central) and  $\tilde{H}_m^i := \frac{H_m^i}{[m]_{q_i}}$ , which satisfy

$$[\tilde{H}_m^i, \tilde{H}_n^j] = \delta_{m+n,0} \frac{[a_{ij}m]_{q_i}}{m[m]_{q_i}} \cdot \frac{[mc]_{q_j}}{[n]_{q_j}} \quad (m, n \in \mathbb{Z}^\times; i, j \in [1, r]).$$

Note that the right hand side is well defined in  $\mathbb{C}(q)$ . Let  $\mathcal{U}_\epsilon(\tilde{\mathfrak{H}})$  be the specialisation of  $\mathcal{U}_q(\tilde{\mathfrak{H}})$ . It is well-defined at the root of unity  $\epsilon$ . The right hand side is also well defined at the root of unity.  $\tilde{H}_{m\ell'}^i$  are not central in  $\mathcal{U}_\epsilon(\tilde{\mathfrak{H}})$ : the centre of  $\mathcal{U}_\epsilon(\tilde{\mathfrak{H}})$  is *not* enlarged at the root of unity  $\epsilon$ . I thank Tetsuji Miwa for giving me this result, about this Lusztig-like form  $\mathcal{U}_\epsilon(\tilde{\mathfrak{H}})$  of  $\mathcal{U}_\epsilon(\mathfrak{H})$ .

### 8.3. The centre of $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ at a root of unity

Let  $\ell$  be an *odd* integer, such that  $\ell > d_i$  ( $\forall i \in [0, r]$ ) and fix  $\epsilon$  to be an odd primitive  $\ell$ -th root of unity ( $\epsilon = e^{2\pi i/\ell}$ ). Then the following fact about the Cartan subalgebra generators is well known.

LEMMA 8.3.1. *The elements  $\{K_i^{\pm\ell}, D^{\pm\ell} \mid i \in [1, r]\}$  are central in  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ .*

**8.3.2.** Recall from chapter 4 the following result concerning the centre of  $\mathcal{U}_\epsilon(\mathfrak{g})$  at the root of unity  $\epsilon$ , which is a special case of corollary 4.8.5.

LEMMA 8.3.3. *The elements  $e_i^\ell$  and  $f_i^\ell$  ( $i \in [1, r]$ ) lie in the centre of  $\mathcal{U}_\epsilon(\mathfrak{g})$  at the root of unity  $\epsilon$ .*

PROPOSITION 8.3.4. *At the root of unity  $\epsilon$ , the elements*

$$\{e_i^\ell, f_i^\ell \mid i \in [0, r]\}$$

*lie in the centre of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ .*

PROOF. The proposition follows from lemmas 4.8.1 and 4.8.2 with  $(\mathfrak{g}, (a_{ij})_{i,j \in [1, r]})$  replaced by  $(\hat{\mathfrak{g}}, (a_{ij})_{i,j \in [0, r]})$ .  $\square$

**PROPOSITION 8.3.5.** *Let  $\{E_m^{\pm, \beta_k} \mid m \in \mathbb{Z}, k \in [1, n]\}$  be the basis of the real root vectors of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  introduced in 7.5.7. The elements in the following set are in the centre of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  at the root of unity  $\epsilon$*

$$\{(E_m^{+, \beta_k})^\ell, (E_m^{-, \beta_k})^\ell \mid m \in \mathbb{Z}, k \in [1, n]\}.$$

**PROOF.** The case  $m = 0$  corresponds to proposition 8.3.4. Consider now one of the generators  $E_0^{+, i}$ . By applying the translation automorphism  $(\hat{T}_{\omega_i^\vee})^m$  ( $m \in \mathbb{Z}^\times$ ) to  $(E_0^{+, i})^\ell$ , it follows that  $(E_{-m}^{+, i})^\ell$  is in the centre of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ . Let  $\{\beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \mid k \in [1, N]\}$  be the set of roots  $R_+$  of  $\mathfrak{g}$  ordered with respect to a reduced expression of  $w_0$  (see 7.5.7). Then  $(E_m^{+, \beta_k})^\ell = T_{i_1} \cdots T_{i_{k-1}}(E_m^{+, i_k})^\ell$  lies in the centre of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ . The result for  $E_m^{-, \beta_k}$  follows by applying the anti-automorphism  $\omega$  and the proposition is proved.  $\square$

**8.3.6.** Let  $\mathcal{Z}_\epsilon$  denote the centre of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ . Let  $z_i^{\pm 1} := K_i^{\pm \ell}$  and  $z_D^{\pm 1} := D^{\pm \ell}$ . Denote by  $\mathcal{Z}_0^0$  the subalgebra of  $\mathcal{U}_\epsilon(\hat{\mathfrak{h}})$  generated by  $\{z_i^{\pm 1}, z_D^{\pm 1} \mid i \in [1, r]\}$ .

For  $\alpha \in R_+$  and  $m \in \mathbb{Z}$ , define  $x_{\alpha+m\delta} := (E_m^{+, \alpha})^\ell$  and  $x_{-\alpha+m\delta} := (E_m^{-, \alpha})^\ell$ . Let  $\mathcal{Z}_0^+$  be the subalgebra of  $\mathcal{U}_\epsilon(\hat{\mathfrak{n}}_+)$  generated by

$$\{x_\alpha, x_{\beta+m\delta}, H_{m\ell}^i \mid \alpha \in R_+, \beta \in R, m \in \mathbb{Z}_{>0}, i \in [1, r]\}.$$

Let  $\mathcal{Z}_0^-$  be the subalgebra of  $\mathcal{U}_\epsilon(\hat{\mathfrak{n}}_-)$  generated by

$$\{x_{-\alpha}, x_{\beta+m\delta}, H_{m\ell}^i \mid \alpha \in R_+, \beta \in R, m \in \mathbb{Z}_{<0}, i \in [1, r]\}.$$

Let  $\mathcal{Z}_0^{\pm, \text{re}}$  ( $\mathcal{Z}_0^{\pm, \text{im}}$ ) denote the subalgebra of  $\mathcal{Z}_0^\pm$  generated by elements constructed out of real (respectively imaginary) root vectors.

Finally define  $\mathcal{Z}_0$  to be the subalgebra of  $\mathcal{Z}_\epsilon$  generated by  $\mathcal{Z}_0^+$ ,  $\mathcal{Z}_0^0$  and  $\mathcal{Z}_0^-$ . Then  $\mathcal{Z}_0 = \mathcal{Z}_0^- \otimes \mathcal{Z}_0^0 \otimes \mathcal{Z}_0^+$ .

**LEMMA 8.3.7.**  *$\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  is infinite dimensional over  $\mathcal{Z}_0$  at the root of unity  $\epsilon$ .*

**PROOF.** The imaginary root vectors  $H_m^i$  ( $m \in \mathbb{Z} \setminus \ell\mathbb{Z}$ ,  $i \in [1, r]$ ) are not in the centre of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  and do not generate central elements. Therefore as a free module over its centre,  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  has basis elements with arbitrarily high powers of  $H_m^i$  and is infinite dimensional.  $\square$

## 8.4. Representations

**8.4.1.** Let  $\hat{P}$  and  $\hat{Q}$  be the weight and root lattices of  $\hat{\mathfrak{g}}$ . Let  $(\cdot, \cdot) : \hat{P} \times \hat{Q} \rightarrow \mathbb{Z}$  be the bilinear form introduced in 7.3.3.

Let  $\lambda = \sum_{i \in [0, r]} \lambda_i \omega_i \in \hat{P}$  be a weight of  $\hat{\mathfrak{g}}$  and  $m \in \mathbb{Z}$ . Denote by  $M(\lambda, m)$  the Verma module over  $\mathcal{U}_q(\hat{\mathfrak{g}})$  generated by a vector  $v_\lambda$  such that

$$\begin{aligned} \mathcal{U}_q(\hat{\mathfrak{n}}_+) \cdot v_\lambda &= 0, \\ \mathcal{U}_q(\hat{\mathfrak{n}}_-) \cdot v_\lambda &= M(\lambda, m), \\ k_i \cdot v_\lambda &= q_i^{\lambda_i} v_\lambda \equiv q^{(\lambda, \alpha_i)} v_\lambda, \\ D \cdot v_\lambda &= q^m v_\lambda. \end{aligned}$$

Of course every highest weight  $\mathcal{U}_q(\hat{\mathfrak{g}})$ -module with highest weight  $(\lambda, m)$ , is a quotient of  $M(\lambda, m)$ . There exists a unique maximal proper submodule  $M'$  of  $M(\lambda, m)$  and therefore the quotient  $L(\lambda, m) := M(\lambda, m)/M'$  is the unique irreducible highest weight  $\mathcal{U}_q(\hat{\mathfrak{g}})$ -module with height weight  $(\lambda, m)$ .

**8.4.2.** As in the classical case [Kac90, 9.2], the Verma module  $M(\lambda, m)$  can be constructed as a quotient of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  by an left ideal  $J(\lambda, m)$  in  $\mathcal{U}_q(\hat{\mathfrak{g}})$

$$J(\lambda, m) = \mathcal{U}_q(\hat{\mathfrak{g}}) \left( \mathcal{U}_q(\hat{\mathfrak{n}}_+) + \sum_{i \in [0, r]} \mathbb{C}(q)((k_i)^{\pm 1} - q_i^{\pm \lambda_i}) + \mathbb{C}(q)(D^{\pm 1} - q^{\pm m}) \right).$$

The quotient  $\mathcal{U}_q(\hat{\mathfrak{g}})/J(\lambda, m) \simeq M(\lambda, m)$ .

**8.4.3. The level.** Let  $C$  be the canonical central element of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  (see 7.2.11). Define the level  $c$  of the  $\mathcal{U}_q(\hat{\mathfrak{g}})$  Verma module  $M(\lambda, m)$  by  $C \cdot v_\lambda = q^c v_\lambda$ .

LEMMA 8.4.4. *The level  $c$  of the  $\mathcal{U}_q(\hat{\mathfrak{g}})$  Verma module  $M(\lambda, m)$  is given by*

$$c = \sum_{i \in [0, r]} a_i(\lambda, \alpha_i).$$

Note that  $c \in \mathbb{Z}$ .

PROOF.

$$\begin{aligned} C \cdot v_\lambda &= \prod_{i \in [0, r]} k_i^{a_i} v_\lambda \\ &= \prod_{i \in [0, r]} q_i^{a_i \langle \lambda, \alpha_i^\vee \rangle} v_\lambda. \end{aligned}$$

□

**8.4.5.** Let  $V$  be a highest weight  $\mathcal{U}_q(\hat{\mathfrak{g}})$ -module with highest weight  $(\lambda, m)$ . Let  $\mu \in \hat{Q}_+$ . Define the subspace  $V_\mu$  of  $V$  as

$$V_\mu := \{v \in V \mid k_i \cdot v = q^{(\lambda - \mu, \alpha_i)} v\}.$$

Then  $V$  admits the following weight space decomposition ( $\hat{Q}_+$ -gradation)

$$V = \bigoplus_{\mu \in \hat{Q}_+} V_\mu,$$

and  $V$  is said to be  $\mathcal{U}_q(\hat{\mathfrak{h}})$ -diagonalisable.

**8.4.6.** Integrable modules of  $\mathcal{U}_q(\hat{\mathfrak{g}})$  can be constructed in complete analogy with integrable modules of  $\mathcal{U}_q(\mathfrak{g})$  (see 4.7.6) by quotienting  $\mathcal{U}_q(\hat{\mathfrak{g}})$  by a suitable left ideal.

## 8.5. Representations at a root of unity

**8.5.1.** Let  $M(\lambda, m)$  be a  $\mathcal{U}_q(\hat{\mathfrak{g}})$  Verma module. Denote by  $M_{\mathcal{A}}(\lambda, m)$  the  $\mathcal{U}_{\mathcal{A}}(\hat{\mathfrak{g}})$ -submodule of  $M(\lambda, m)$ . Denote by  $M_\epsilon(\lambda, m)$  the  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  Verma module, which is the specialisation of  $M_{\mathcal{A}}(\lambda, m)$  at  $q = \epsilon$ .

**8.5.2. Diagonal modules.** Consider the  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  Verma module  $M_\epsilon(\lambda, m)$  at the root of unity  $\epsilon$ , generated by  $v_\lambda$ . The action of the elements in  $\mathcal{Z}_0^-$  on  $v_\lambda$  generates singular vectors in  $M_\epsilon(\hat{\mathfrak{g}})$ .

Define the diagonal module  $\bar{M}_\epsilon(\lambda, m)$  of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  to be

$$\bar{M}_\epsilon(\lambda, m) := M_\epsilon(\lambda, m) / (\mathcal{U}_\epsilon(\hat{\mathfrak{g}})\mathcal{Z}_0^- \cdot v_\lambda).$$

The module  $\bar{M}_\epsilon(\lambda, m)$  is infinite dimensional. The  $\mathcal{U}_\epsilon(\mathfrak{g})$ -submodule of  $M_\epsilon(\lambda, m)$  generated by its highest weight vector  $v_\lambda$  is finite dimensional and coincides with the diagonal  $\mathcal{U}_\epsilon(\mathfrak{g})$ -module constructed in 4.8.6.

**8.5.3. Triangular modules.** Let  $\lambda \in \hat{P}$  and let  $\nu$  be an algebra homomorphism  $\nu : \mathcal{Z}_0^- \rightarrow \mathbb{C}$ . Define the following left ideal in  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$

$$\mathcal{I}_{\text{tri}}(\lambda, m, \nu) := \mathcal{U}_\epsilon(\hat{\mathfrak{g}}) \left( \sum_{i \in [0, r]} e_i + \sum_{i \in [0, r]} (k_i - q^{(\lambda, \alpha_i)}) + (D - q^m) + \sum_{y \in \mathcal{Z}_0^-} (y - \nu(y)) \right).$$

Then define the *triangular module*  $\bar{M}_\epsilon(\lambda, m, \nu)$  over  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  to be

$$\bar{M}_\epsilon(\lambda, m, \nu) := \mathcal{U}_\epsilon(\hat{\mathfrak{g}}) / \mathcal{I}_{\text{tri}}(\lambda, m, \nu).$$

**8.5.4.** The triangular module  $\bar{M}_\epsilon(\lambda, m, 0)$  ( $0 : \mathcal{Z}_0^- \rightarrow 0$ ) coincides with the diagonal module  $\bar{M}_\epsilon(\lambda, m)$ . This is a nilpotent representation of  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  since the Chevalley generators act on it nilpotently.

Let  $\alpha \in R_+$ . When  $\nu(y_\alpha) \neq 0$ , then  $f_\alpha$  acts cyclicly in  $\bar{M}_\epsilon(\lambda, m, \nu)$ . In the case that  $\nu(\mathcal{Z}_0^{\text{re}}) \neq 0$ , the module is called semicyclic (semiperiodic).

**8.5.5. Central characters.** Let  $V$  be an irreducible  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ -module at the root of unity  $\epsilon$ , then each central element  $x \in \mathcal{Z}_\epsilon$  acts on  $V$  as a scalar  $\chi(x)$ . The map  $\chi : \mathcal{Z}_\epsilon \rightarrow \mathbb{C}$  is the central character of the representation. Note in particular that  $\chi^\pi(z_i), \chi^\pi(z_D) \in \mathbb{C}^\times$ .

If a representation is a diagonal  $\mathcal{U}_\epsilon(\mathfrak{g})$ -module, then  $\chi : \mathcal{Z}_0^{\pm, \text{re}} \rightarrow 0 \in \mathbb{C}$ . For a triangular module the central character maps  $\chi : \mathcal{Z}_0^{+, \text{re}} \rightarrow 0$ . In a completely cyclic (periodic) module  $\chi(x_\beta), \chi(y_\beta) \in \mathbb{C}^\times$  ( $\beta \in \hat{R}_+^{\text{re}}$ ).

**PROPOSITION 8.5.6.** *Let  $V$  be a indecomposable highest weight  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  module at the root of unity with nonzero level  $c$  such that  $\epsilon^c \neq 1$ .  $V$  is infinite dimensional.*

**PROOF.** This follows since every Heisenberg  $\mathcal{U}_\epsilon(\mathfrak{h})$  submodule of  $V$  will be infinite dimensional.  $\square$

**LEMMA 8.5.7.** *Let  $V$  be an indecomposable  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  module at the root of unity.  $V$  has a finite number of weight spaces.*

**8.5.8.** Let  $\nu : \mathcal{Z}_0 \rightarrow \mathbb{C}$ , such that  $\nu(z_i), \nu(z_D) \in \mathbb{C}^\times$ . Define  $P_\epsilon$  to be

$$P_\epsilon(\nu) := \mathcal{U}_\epsilon(\hat{\mathfrak{g}}) / (\mathcal{U}_\epsilon(\hat{\mathfrak{g}}) \sum_{z \in \mathcal{Z}_0} z - \nu(z)).$$

$P_\epsilon(\nu)$  has a natural  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$  module structure. Every  $\mathcal{U}_\epsilon(\hat{\mathfrak{g}})$ -module  $V$  generated by a single vector, with central character  $\chi$ , is equivalent to some quotient module of  $P_\epsilon(\chi)$ .

In the case  $\nu(\mathcal{Z}_0^\pm) = 0$  the module  $P_\epsilon$  and its quotients are called nilpotent. In the case that  $\nu(\mathcal{Z}_0^{+, \text{re}}) = 0$  ( $\nu(\mathcal{Z}_0^{-, \text{re}}) = 0$ ) and  $\nu(y_\beta) \neq 0$  ( $\nu(x_\beta) \neq 0$ ) ( $\beta \in \hat{R}_+^{\text{re}}$ ) the module  $P_\epsilon$  and its quotients are called completely semicyclic (semiperiodic). In the case that  $\nu(x_\beta) \neq 0$  and  $\nu(y_\beta) \neq 0$  ( $\beta \in \hat{R}_+^{\text{re}}$ ) the module  $P_\epsilon$  and its quotients are called completely cyclic (periodic).



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